Math 1030 Workshop #4 (July 23, 2009) **Name: <u>ANSWER KEY</u>**

1. Use the limit definition of the derivative to find $f'(x)$ if...
[You use should the rules we learned to double check your answer.]

(a) $f(x) = -x^2 + 3x - 1$

Notice that $f(x+h) = -(x+h)^2 + 3(x+h) - 1$ so..

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\big(-(x+h)^2 + 3(x+h) - 1\big) - \big(-x^2 + 3x - 1\big)}{h} \\ &= \lim_{h\to 0} \frac{-x^2 - 2xh - h^2 + 3x + 3h - 1 + x^2 - 3x + 1}{h} \\ &= \lim_{h\to 0} \frac{-2xh - h^2 + 3h}{h} \\ &= \lim_{h\to 0} \frac{h(-2x - h + 3)}{h} \\ &= \lim_{h\to 0} -2x - h + 3 \\ &= -2x + 3 \end{aligned}$$

So $f'(x) = -2x + 3$ (which is exactly what our "rules" tell us.)

(b) $f(x) = \dfrac{2}{(x-1)^2}$

Notice that $f(x+h) = \dfrac{2}{((x+h)-1)^2}$ so..

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2}{(x+h-1)^2} - \frac{2}{(x-1)^2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{2(x-1)^2}{(x-1)^2(x+h-1)^2} - \frac{2(x+h-1)^2}{(x-1)^2(x+h-1)^2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{2(x-1)^2 - 2(x+h-1)^2}{(x-1)^2(x+h-1)^2}}{h/1} \\ &= \lim_{h\to 0} \frac{2x^2 - 4x + 2 - 2x^2 + 4x - 2 - 4xh + 4h - 2h^2}{h(x-1)^2(x+h-1)^2} \\ &= \lim_{h\to 0} \frac{-4xh + 4h - 2h^2}{h(x-1)^2(x+h-1)^2} \\ &= \lim_{h\to 0} \frac{h(-4x + 4 - 2h)}{h(x-1)^2(x+h-1)^2} \\ &= \lim_{h\to 0} \frac{-4x + 4 - 2h}{(x-1)^2(x+h-1)^2} \\ &= \frac{-4x + 4}{(x-1)^2(x-1)^2} \\ &= \frac{-4(x-1)}{(x-1)^4} \\ &= \frac{-4}{(x-1)^3} \end{aligned}$$

So $f'(x) = \dfrac{-4}{(x-1)^3}$ (To double check notice that $f(x) = 2(x-1)^{-2}$. Using the generalized power rule, we get $f'(x) = 2(-2)(x-1)^{-2-1}(1) = -4(x-1)^{-3}$ which matches our answer.)

2. Find the equation of the line tangent to the graph of $y = f(x)$ at $x = x_0$ if...

   (a) $f(x) = -x^2 + 3x - 1$ and $x_0 = 1$

   Notice that if $x_0 = 1$, then $f(x_0) = f(1) = -1^2+3(1)-1 = 1$, so our line passes through the point $(x, y) = (1, 1)$. Next, we need to know the slope, which is given by the derivative. $f'(x) = -2x+3$ and so our slope is $f'(x_0) = f'(1) = -2(1) + 3 = 1$. Using "point-slope" we find $y - 1 = 1(x - 1)$ so that...

   Answer: $y = x$
   [Try typing "`plot -x^2+3x-1 and x for x=-2..2`" into Wolfram Alpha to see a nice plot.]

   (b) $f(x) = \dfrac{2}{(x-1)^2}$ and $x_0 = 0$

   Notice that if $x_0 = 0$, then $f(x_0) = f(0) = 2/(0-1)^2 = 2/(-1)^2 = 2$, so our line passes through the point $(x, y) = (1, 2)$. Next, we need to know the slope, which is given by the derivative. $f'(x) = -4(x-1)^{-3}$ and so our slope is $f'(x_0) = f'(0) = -4(0-1)^3 = -4(-1)^3 = -4(-1) = 4$. Using "point-slope" we find $y - 2 = 4(x - 0)$ so that...

   Answer: $y = 4x + 2$
   [Try typing "`plot 2/(x-1)^2 and 4x+2 for x=-2..1`" into Wolfram Alpha to see a nice plot.]

3. Compute the derivative of each of the following functions. Please simplify your answers.

   (a) $y = 3e^x - 5\ln(x) + \sqrt{x} + \frac{1}{x^3} + 3x + 2$ $(= 3e^x - 5\ln(x) + x^{1/2} + x^{-3} + 3x + 2)$

   $$y' = 3e^x - 5\frac{1}{x} + \frac{1}{2}x^{-1/2} - 3x^{-4} + 3 = 3e^x - \frac{5}{x} + \frac{1}{2\sqrt{x}} - \frac{3}{x^4} + 3$$

   (b) $y = \ln(x)e^{-x}$ [Remember to use the chain rule on e^{-x}.]

   $$y' = \frac{1}{x}e^{-x} + \ln(x)e^{-x}(-1) = \frac{e^x}{x} - e^{-x}\ln(x)$$

   (c) $y = \dfrac{x^2+3x-5}{xe^x}$ [Quotient rule + Product rule for the denominator]

   $$\begin{aligned} y' &= \frac{(2x+3)xe^x - (x^2+3x-5)(e^x + xe^x)}{(xe^x)^2} \\ &= \frac{2x^2e^x + 3xe^x - x^2e^x - 3xe^x + 5e^x - x^3e^x - 3x^2e^x + 5xe^x}{x^2e^{2x}} \\ &= \frac{e^x(-x^3 - 2x^2 + 5x + 5)}{x^2e^{2x}} \\ &= \frac{-x^3 - 2x^2 + 5x + 5}{x^2e^x} \end{aligned}$$

   (d) $y = (x\ln(x) + 5)^{100}$ [Generalized power rule + Product rule]

   $$y' = 100(x\ln(x)+5)^{99}\left(1\ln(x) + x\frac{1}{x} + 0\right) = 100(x\ln(x)+5)^{99}(1 + \ln(x))$$

   (e) $y = \ln\left((x^2+1)e^{-3x}\right)$ $(= \ln(x^2+1) + \ln(e^{-3x}) = \ln(x^2+1) - 3x)$ [Chain rule on first term]

   $$y' = \frac{1}{x^2+1}(2x) - 3 = \frac{2x}{x^2+1} - 3$$