Math 1030 Workshop #5 (Due: July 29^{th}, 2011) **Name:** ANSWER KEY

Please turn in a paper copy and **SHOW YOUR WORK!**

1. Consider the function $P(q) = \begin{cases} -2q^2 - 5q + 1 & q < 0 \\ -2q^3 + q^2 + 4q + 2 & 0 \le q \le 2 \\ (q^2 - 5q + 9)e^{-(q-3)^2/10} & q > 2 \end{cases}$

   Be careful! Wolfram Alpha has a hard time interpreting commands applied to this function. You may want to deal with the function one piece at a time.

   $P(q)$ defined in ALPHA:

   `piecewise[{{-2q^2-5q+1, q<0}, {-2q^3+q^2+4q+2, 0<=q<=2}, {(q^2-5q+9)e^(-(q-3)^2/10), q>2}}]` ✓

   We are (eventually) asked to find the minimum and maximum when $-2 \le x \le 8$, so let's plot this function on that interval:

   `plot piecewise[{{-2q^2-5q+1, q<0}, {-2q^3+q^2+4q+2, 0<=q<=2}, {(q^2-5q+9)e^(-(q-3)^2/10), q>2}}] on [-2,8]` ✓

   We need to find critical points. These are points where the derivative is undefined or equal to zero. First of all, the derivative cannot be defined at $q = 0$ and $q = 2$ since $P(q)$ is discontinuous at these points (see the graph) – this isn't surprising since we usually have a critical point where our piecewise defined function switches formulas.

   Next, we would like to find the derivative, but ALPHA won't differentiate piecewise defined functions (at least not easily). So we will deal with our function one piece at a time.

   `derivative -2q^2-5q+1` ✓

   We find that this derivative has a root at $q = -5/4 = -1.25$. Since $-1.25 < 0$, this point occurs in the valid range for the first formula (i.e. $q < 0$). So $q = -1.25$ is a critical point of $P(q)$.

   `derivative -2q^3+q^2+4q+2` ✓

   We find that this derivative has 2 roots: $q = -2/3$ and $q = 1$. However, $q = -2/3 < 0$ and so this point is outside the range where the second formula is used (i.e. $0 \le q \le 2$). On the other hand $0 \le q = 1 \le 2$, so $q = 1$ is a critical point of $P(q)$.

   `derivative (q^2-5q+9)e^(-(q-3)^2/10)` ✓

   We find that this derivative has 3 roots: $q = 0.156596$, 2.30677, and 5.53663. However, $q = 0.156596 < 2$ so it is out of bounds for the third formula. Whereas the other 2 points are bigger than 2 and so are critical points of $P(q)$.

   (a) Find all of the critical points of $P(q)$. $q =$ <u>-1.25, 0, 1, 2, 2.30677, and 5.53663</u>

   When we look at the graph is it obvious that the minimum occurs at the critical point $q = 2$ (the end of the second formula's graph) and the maximum occurs at the peak of the third formula's graph somewhere near $q = 5$, so this must be the last critical point $q = 5.53663$. Alternatively, we could plug in all of the critical points along with the end points $q = -2$ and $q = 8$ and see which values are the biggest and smallest — this is essentially what we'd need to do if we didn't have a reliable graph. [*Note:* Technically, we would also need to check right & left hand limits at each point of discontinuity. The extreme value theorem doesn't apply directly since our function isn't continuous. However, we won't worry about these technicalities and just rely on the graph.]

   Plugging in $q = 2$ we find:

   `piecewise[{{-2q^2-5q+1, q<0}, {-2q^3+q^2+4q+2, 0<=q<=2}, {(q^2-5q+9)e^(-(q-3)^2/10), q>2}}] where q=2` ✓

   or just plugging directly into the second formula (which gets used when $0 \le q \le 2$) we find:

   `-2q^3+q^2+4q+2 where q=2` ✓

   So $P(2) = -2$. Next, plugging in $q = 5.53663$ we find:

   `piecewise[{{-2q^2-5q+1, q<0}, {-2q^3+q^2+4q+2, 0<=q<=2}, {(q^2-5q+9)e^(-(q-3)^2/10), q>2}}] where q=5.53663` ✓

   or just plugging into the third formula (which gets used when $q > 2$) we find:

   `(q^2-5q+9)e^(-(q-3)^2/10) where q=5.53663` ✓ So $P(5.53663) = 6.29055$.

(b) Restricting our attention to the interval $[-2, 8]$...

The maximum value of $P(q)$ is 6.29055. This occurs when $q =$ 5.53663.

The minimum value of $P(q)$ is -2. This occurs when $q =$ 2.

2. Carl manages a motel in Spruce Pine. He needs to keep a TV in each room in the motel and has found nice TV's which he can purchase for \$260. In addition, Carl has noticed that he spends an average of \$10 repairing TV's during their first year of operation, and then spends an average of \$25 repairing TV's during their second year of operation. Model the average annual cost of a TV using a function of the form: $A(t) = \frac{C}{t} + Rt^r$ where C is the cost of purchasing the TV and Rt^r models the repair costs.

Use the facts $Rt^r = 10$ when $t = 1$ and $Rt^r = 17.5 \left(= \frac{10+25}{2}\right)$ when $t = 2$ to find R and r [Keep r out to 6 decimal places].

We have that $Rt^r = 10$ when $t = 1$ so $R(1^r) = 10$. Using ALPHA or just the fact that $1^r = 1$ (for any r) we get that $R = 10$. Next, $Rt^r = 17.5$ when $t = 2$ so that $10(2^r) = 17.5$. ALPHA will happily solve this for us.

`10(2^r)=17.5` ✓

Thus $r = 0.807355$

$$A(t) = \frac{260}{t} + 10t^{0.807355}$$

We need to find the minimum value of $A(t)$. So we take it's derivative and find critical points.

`derivative 260/t + 10t^0.807355` ✓

We can see that $A'(t)$ is not defined at $t = 0$ (but we should only consider $t > 0$). We have a root at $t = 6.82839$. Looking at the plot where $1 \leq t \leq 10$

`plot 260/t + 10t^0.807355 on [1,10]` ✓

we can see that $t = 6.82839$ is the minimum. Plugging this in...

`260/t + 10t^0.807355 where t=6.82839` ✓

We find that $C(6.82839) = \$85.24$. Also, $t = 6.82839$ is 6 years and $0.82839 \times 12 = 9.94068$ months.

Carl should replace his TV's every 6 years and 10 months.

If he does this, his average annual cost (per TV) will be \$ 85.24.

3. Cindy sells locally-produced, scented candles in her store. She pays \$8 per candle and has found that her average inventory costs are \$0.25 per candle per year (base inventory costs on average inventory making all of the standard assumptions). Suppose Cindy is charged \$10 every time she places an order. Let $C(x)$ be Cindy's annual cost function.

(a) If Cindy sells 50 candles each year, $C(x) = 8(50) + 10\left(\frac{50}{x}\right) + 0.25\left(\frac{x}{2}\right)$

`plot 8(50)+10(50/x)+0.25(x/2) on [1,50]` ✓

`derivative 8(50)+10(50/x)+0.25(x/2)` ✓

We have critical points at $x = 0, \pm 63.2456$. None of these points are within the bounds of this part of the problem (i.e. $1 \leq x \leq 50$), so the minimum must occur at an endpoint. From the plot (and common sense) we know that the minimum occurs when $x = 50$.

`8(50)+10(50/x)+0.25(x/2) where x=50` ✓

Her **ideal** EOQ is 50 candles and her **ideal** minimum annual cost is \$416.25.

[*Note:* This is also a practical solution — not just ideal.]

(b) If Cindy sells 100 candles each year, $C(x) = 8(100) + 10\left(\frac{100}{x}\right) + 0.25\left(\frac{x}{2}\right)$

`plot 8(100)+10(100/x)+0.25(x/2) on [1,100]` ✓

`derivative 8(100)+10(100/x)+0.25(x/2)` ✓

We have critical points at $x = 0, \pm 89.4427$. Only $x = 89.4427$ lies within the bounds $1 \leq x \leq 100$. So we should plug in $x = 1$, 89.4427, and 100 to find out where the maximum occurs (and its value). Although common sense tells us the answer is **not** $x = 1$. We should also know by now (from every other EOQ problem we've done) that the minimum is going to occur at the critical point (when it's in bounds).

`8(100)+10(100/x)+0.25(x/2) where x=[1,89.4427,100]` ✓

Her **ideal** EOQ is <u>89.4427 candles</u> and her **ideal** minimum annual cost is <u>$822.36</u>.

(c) Suppose that Cindy sells 100 candles per year and gets a discount if she places a large order. For orders of 20 or more candles, she pays $6.50 each. However, her shipping costs increase to $12 for a large shipment. (Inventory stays the same.)

$$C(x) = \begin{cases} 8(100) + 10\left(\frac{100}{x}\right) + 0.25\left(\frac{x}{2}\right) & x < 20 \\ 6.5(100) + 12\left(\frac{100}{x}\right) + 0.25\left(\frac{x}{2}\right) & x \geq 20 \end{cases}$$

```
plot piecewise[{{8(100)+10(100/x)+0.25(x/2), x<20},
                              {6.5(100)+12(100/x)+0.25(x/2), x>=20}}] on [1,100] ✓
```

From the graph we can see that the minimum occurs when $x \geq 20$. So we discard the first formula and focus on the second.

`derivative 6.5(100)+12(100/x)+0.25(x/2)` ✓

We find that this function has 3 critical points at $x = 0$ and ± 97.9796. Only $x = 97.9796$ is within this formulas bounds (i.e. $20 \leq x \leq 100$). So our minimum is going to occur either at $x = 97.9796$ or $x = 100$.

`6.5(100)+12(100/x)+0.25(x/2) where x=[97.9796, 100]` ✓

The critical point gives the smaller amount, so we find that...

Cindy's **ideal** EOQ is <u>97.9796 candles</u>. Her **ideal** minimum annual cost is <u>$674.50</u>.