Math 1030 Workshop #4 **Name:** ANSWER KEY

DUE: Tuesday, July 23$^{\text{rd}}$ Please turn in a paper copy and **SHOW YOUR WORK!**

1. Use the limit definition of the derivative to find $f'(x)$ if...
[You use should the rules we learned to double check your answer.]

(a) $f(x) = x^3 + x^2 + 1$ Note that $f\left(\boxed{x+h}\right) = \left(\boxed{x+h}\right)^3 + \left(\boxed{x+h}\right)^2 + 1$. Also, $(x+h)^3 = (x+h)(x+h)(x+h) = (x+h)(x^2+2xh+h^2) = x^3+2x^2h+xh^2+hx^2+2xh^2+h^3 = x^3+3x^2h+3xh^2+h^3$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\left((x+h)^3+(x+h)^2+1\right)-\left(x^3+x^2+1\right)}{h} \\ &= \lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3+\cancel{x^2}+2xh+h^2+\cancel{1}-\cancel{x^3}-\cancel{x^2}-\cancel{1}}{h} \\ &= \lim_{h\to 0}\frac{\cancel{h}\left(3x^2+3xh+h^2+2x+h\right)}{\cancel{h}} = 3x^2+3x(0)+0^2+2x+0 \end{aligned}$$

Answer: $f'(x) = 3x^2 + 2x$ (This is easily verified using our formulas for differentiation.)

(b) $f(x) = \dfrac{1}{x^2-2}$ Note that $f\left(\boxed{x+h}\right) = \dfrac{1}{\left(\boxed{x+h}\right)^2 - 2}$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{(x+h)^2-2}-\frac{1}{x^2-2}}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2-2}\cdot\frac{x^2-2}{x^2-2}-\frac{1}{x^2-2}\cdot\frac{(x+h)^2-2}{(x+h)^2-2}}{h} = \lim_{h\to 0}\frac{\frac{x^2-2}{((x+h)^2-2)(x^2-2)}-\frac{(x+h)^2-2}{(x^2-2)((x+h)^2-2)}}{h} \\ &= \lim_{h\to 0}\frac{\frac{(x^2-2)-((x+h)^2-2)}{((x+h)^2-2)(x^2-2)}}{\frac{h}{1}} = \lim_{h\to 0}\frac{\cancel{x^2}-\cancel{2}-\cancel{x^2}-2xh-h^2+\cancel{2}}{h((x+h)^2-2)(x^2-2)} \\ &= \lim_{h\to 0}\frac{\cancel{h}(-2x-h)}{\cancel{h}((x+h)^2-2)(x^2-2)} = \frac{-2x-0}{((x+0)^2-2)(x^2-2)} = \frac{-2x}{(x^2-2)^2} \end{aligned}$$

This result can be verified using either the quotient rule: $f'(x) = \dfrac{0(x^2-2)-1(2x)}{(x^2-2)^2}$ or using a bit of algebra: $f(x) = (x^2-2)^{-1}$ and then the chain rule: $f'(x) = (-1)(x^2-2)^{-2}(2x)$. Anyway we go about it we find that...

Answer: $f'(x) = \boxed{\dfrac{-2x}{(x^2-2)^2}}$

2. Find the equation of the line tangent to the graph of $y = f(x)$ at $x = x_0$ if...

(a) $f(x) = x^3 + x^2 + 1$ and $x_0 = -2$

To find the equation of the tangent line we need a point and a slope. We get our point by plugging $x_0 = -2$ into $f(x)$ and we get our slope by plugging $x_0 = -2$ into $f'(x)$.
First, $f(-2) = (-2)^3 + (-2)^2 + 1 = -8 + 4 + 1 = -3$. So our line passes through $(x, y) = (-2, -3)$.
Next, $f'(x) = 3x^2 + 2x$ so $f'(-2) = 3(-2)^2 + 2(-2) = 12 - 4 = 8$. So our line has slope $m = 8$.
Finally, using point-slope we get $y - (-3) = 8(x - (-2))$ so $y + 3 = 8(x + 2)$ and so $y + 3 = 8x + 16$ and thus...

Answer: The equation of the tangent line is $\boxed{y = 8x + 13}$.

(b) $f(x) = \dfrac{1}{x^2-2}$ and $x_0 = 1$

To find the equation of the tangent line we need a point and a slope. We get our point by plugging $x_0 = 1$ into $f(x)$ and we get our slope by plugging $x_0 = 1$ into $f'(x)$.
First, $f(1) = \dfrac{1}{1^2-2} = \dfrac{1}{-1} = -1$. So our line passes through $(x, y) = (1, -1)$.

Next, $f'(x) = \dfrac{-2x}{(x^2-2)^2}$ so $f'(1) = \dfrac{-2(1)}{(1^2-2)^2} = \dfrac{-2}{(-1)^2} = -2$. So our tangent has slope $m = -2$.
Finally, using point-slope we get $y - (-1) = -2(x-1)$ so $y + 1 = -2x + 2$ and thus...

Answer: The equation of the tangent line is $\boxed{y = -2x + 1}$.

3. Compute the derivative of each of the following functions. **Please simplify your answers.**

(a) $y = \sqrt[3]{x} - 12e^x + 4\ln(x) + \frac{1}{x^7} + 9x - 2$

First some algebra: $y = x^{1/3} - 12e^x + 4\ln(x) + x^{-7} + 9x - 2$

$$y' = \boxed{\frac{1}{3}x^{-2/3} - 12e^x + \frac{4}{x} + (-7)x^{-8} + 9}$$

Notes: The derivative of each term follows from either a basic formula or the power rule.

(b) $y = (x^7 + 3)\ln(5x+1)$

$$y' = \boxed{7x^6\ln(5x+1) + (x^7+3)\frac{1}{5x+1}(5)}$$

Notes: Use the product rule with first part $x^7 + 3$ and second part $\ln(5x+1)$. We use the chain rule when differentiating $\ln(5x+1)$ with outside function ln(BLAH) and inside function $5x + 1$. We couldn't simplify $\ln(5x+1)$ since there are 2 terms added together – laws of logs won't help here.

(c) $y = \dfrac{x\ln(x) + 1}{x^2 + 3x + 6}$

$$\begin{aligned} y' &= \frac{\left((1)\ln(x) + x\frac{1}{x}\right)(x^2+3x+6) - (x\ln(x)+1)(2x+3)}{(x^2+3x+6)^2} \\ &= \frac{(\ln(x)+1)(x^2+3x+6) - (x\ln(x)+1)(2x+3)}{(x^2+3x+6)^2} \\ &= \frac{x^2\ln(x) + \cancel{3x\ln(x)} + 6\ln(x) + x^2 + 3x + 6 - 2x^2\ln(x) - \cancel{3x\ln(x)} - 2x - 3}{(x^2+3x+6)^2} \\ &= \boxed{\frac{(6-x^2)\ln(x) + x^2 + x + 3}{(x^2+3x+6)^2}} \end{aligned}$$

Notes: Use the quotient rule. Also, we need the product rule to help take the derivative of $x\ln(x)$. The rest is algebra.

(d) $y = \left(1 + 3e^{x^2}\right)^{11}$

$$y' = 11\left(1 + 3e^{x^2}\right)^{10}\left(3e^{x^2}(2x)\right) = \boxed{66xe^{x^2}\left(1 + 3e^{x^2}\right)^{10}}$$

Notes: Use the chain rule (specifically the "generalized power rule") with outside function BLAH^{11} and inside function $1 + 3e^{x^2}$. To differentiate $3e^{x^2}$ we need the chain rule again. This time $3e^{\text{BLAH}}$ is our outside function and x^2 is our inside function.

(e) $y = \ln\left(\dfrac{7(x^3+1)^5}{e^{-2x}\sqrt{x-2}}\right)$

First some algebra: $y = \ln\left(7(x^3+1)^5\right) - \ln\left(e^{-2x}(x-2)^{1/2}\right)$
$= \ln(7) + \ln\left((x^3+1)^5\right) - \left(\cancel{\ln}(\cancel{e}^{-2x}) + \ln\left((x-2)^{1/2}\right)\right)$
$= \ln(7) + 5\ln(x^3+1) + 2x - \dfrac{1}{2}\ln(x-2)$

$$y' = 0 + 5\frac{1}{x^3+1}(3x^2) + 2 - \frac{1}{2}\cdot\frac{1}{x-2}(1) = \boxed{\frac{15x^2}{x^3+1} + 2 - \frac{1}{2x-4}}$$

Notes: The derivative of $\ln(7)$ is 0 since $\ln(7)$ is a constant (it has no x's in it!). $5\ln(x^3+1)$ can be differentiated using the chain rule the same is true of $\frac{1}{2}\ln(x-2)$.