Math 1030 Workshop #4 **Name:** Answer Key

DUE: Wednesday, July 22^{nd} Please turn in a paper copy and **SHOW YOUR WORK!**

1. Use the limit definition of the derivative to find $f'(x)$ if...
   [You use should the rules we learned to double check your answer.]

   (a) $f(x) = 3x^2 - 5x + 12$

   Note that $f\left(\boxed{x+h}\right) = 3\left(\boxed{x+h}\right)^2 - 5\left(\boxed{x+h}\right) + 12$.
   Also, $3(x+h)^2 = 3(x+h)(x+h) = 3(x^2+2xh+h^2) = 3x^2+6xh+3h^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\big(3(x+h)^2-5(x+h)+12\big)-\big(3x^2-5x+12\big)}{h} \\ &= \lim_{h\to 0}\frac{\cancel{3x^2}+6xh+3h^2-\cancel{5x}-5h+\cancel{12}-\cancel{3x^2}+\cancel{5x}-\cancel{12}}{h} \\ &= \lim_{h\to 0}\frac{\cancel{h}(6x+3h-5)}{\cancel{h}} = 6x+3(0)-5 \end{aligned}$$

   Answer: $f'(x) = 6x - 5$ (This is easily verified using our formulas for differentiation.)

   (b) $f(x) = \dfrac{1}{(x-2)^2}$ Note that $f\left(\boxed{x+h}\right) = \dfrac{1}{\left(\boxed{x+h}-2\right)^2}$. Also, $(x+h-2)^2 = (x+h-2)(x+h-2) = x(x+h-2)+h(x+h-2)-2(x+h-2) = x^2+xh-2x+hx+h^2-2h-2x-2h+4 = x^2+2xh+h^2-4x-4h+4$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{(x+h-2)^2}-\frac{1}{(x-2)^2}}{h} \\ &= \lim_{h\to 0}\frac{\frac{1}{(x+h-2)^2}\cdot\frac{(x-2)^2}{(x-2)^2}-\frac{1}{(x-2)^2}\cdot\frac{(x+h-2)^2}{(x+h-2)^2}}{h} = \lim_{h\to 0}\frac{\frac{(x-2)^2}{(x+h-2)^2(x-2)^2}-\frac{(x+h-2)^2}{((x-2)^2(x+h-2)^2}}{h} \\ &= \lim_{h\to 0}\frac{\frac{(x-2)^2-(x+h-2)^2}{(x+h-2)^2(x-2)^2}}{\frac{h}{1}} = \lim_{h\to 0}\frac{x^2-4x+4-(x^2+2xh+h^2-4x-4h+4)}{h((x+h-2)^2(x-2)^2)} \\ &= \lim_{h\to 0}\frac{\cancel{x^2}-\cancel{4x}+\cancel{4}-\cancel{x^2}-2xh-h^2+\cancel{4x}+4h-\cancel{4}}{h(x+h-2)^2(x-2)^2} = \lim_{h\to 0}\frac{\cancel{h}(-2x-h+4)}{\cancel{h}(x+h-2)^2(x-2)^2} \\ &= \frac{(-2x-0+4)}{(x+0-2)^2(x-2)^2} = \frac{-2(x-2)}{(x-2)^4} = \frac{-2}{(x-2)^3} \end{aligned}$$

   This result can be verified using either the quotient rule: $f'(x) = \dfrac{0(x-2)^2 - 1(2(x-2)^1(1))}{((x-2)^2)^2} = \dfrac{-2(x-2)}{(x-2)^4} = \dfrac{-2}{(x-2)^3}$ or using a bit of algebra: $f(x) = (x-2)^{-2}$ and then the chain rule: $f'(x) = (-2)(x-2)^{-3}(1) = \dfrac{-2}{(x-2)^3}$. Anyway we go about it we find that...

   Answer: $f'(x) = \boxed{\dfrac{-2}{(x-2)^3}}$

2. Find the equation of the line tangent to the graph of $y = f(x)$ at $x = x_0$ if...

   (a) $f(x) = 3x^2 - 5x + 12$ and $x_0 = -1$

   To find the equation of the tangent line we need a point and a slope. We get the point by plugging $x_0 = -1$ into $f(x)$ and we get our slope by plugging $x_0 = -1$ into $f'(x)$.
   First, $f(-1) = 3(-1)^2 - 5(-1) + 12 = 3 + 5 + 12 = 20$. So our line passes through the point $(x, y) = (-1, 20)$.
   Next, $f'(x) = 6x - 5$ so $f'(-1) = 6(-1) - 5 = -11$. So our line has slope $m = -11$.

Finally, using point-slope we get $y - 20 = -11(x - (-1))$ so $y - 20 = -11(x + 1)$ and so $y = -11x - 11 + 20$ and thus...

Answer: The equation of the tangent line is $\boxed{y = -11x + 9}$.

(b) $f(x) = \dfrac{1}{(x-2)^2}$ and $x_0 = 3$

To find the equation of the tangent line we need a point and a slope. We get the point by plugging $x_0 = 3$ into $f(x)$ and we get our slope by plugging $x_0 = 3$ into $f'(x)$.

First, $f(3) = \dfrac{1}{(3-2)^2} = \dfrac{1}{1^2} = 1$. So our line passes through the point $(x, y) = (3, 1)$.

Next, $f'(x) = \dfrac{-2}{(x-2)^3}$ so $f'(3) = \dfrac{-2}{(3-2)^3} = \dfrac{-2}{1^3} = -2$. So our line has slope $m = -2$.

Finally, using point-slope we get $y - 1 = -2(x - 3)$ so $y - 1 = -2x + 6$ and thus...

Answer: The equation of the tangent line is $\boxed{y = -2x + 7}$.

3. Compute the derivative of each of the following functions. **Please simplify your answers.**

(a) $y = \sqrt[5]{x} + 3\ln(x) + 7e^x - \dfrac{11}{x^{10}} - 5x + 123$

First some algebra: $y = x^{1/5} + 3\ln(x) + 7e^x - 11x^{-10} - 5x + 123$.

$$y' = \boxed{\frac{1}{5}x^{-4/5} + \frac{3}{x} + 7e^x + 110x^{-11} - 5}$$

Notes: The derivative of each term follows from either a basic formula or the power rule.

(b) $y = e^{-4x}\ln(x)$

$$y' = \boxed{-4e^{-4x}\ln(x) + \frac{e^{-4x}}{x}}$$

Notes: Use the product rule with first part e^{-4x} and second part $\ln(x)$. We use the chain rule when differentiating e^{-4x} with outside function e^{BLAH} and inside function $-4x$. The second term was: $e^{-4x} \cdot \dfrac{1}{x}$ which we multiplied together in the final answer.

(c) $y = \dfrac{x^2 - 5x + 1}{xe^x}$

$$\begin{aligned}
y' &= \frac{(2x-5)(xe^x) - (x^2-5x+1)((1)e^x + xe^x)}{(xe^x)^2} &&= \frac{2x^2e^x - 5xe^x - (x^2-5x+1)(e^x + xe^x)}{x^2(e^x)^2} \\
&= \frac{2x^2e^x - \cancel{5xe^x} - x^2e^x + \cancel{5xe^x} - e^x - x^3e^x + 5x^2e^x - xe^x}{x^2(e^x)^2} &&= \frac{-x^3e^x + 6x^2e^x - xe^x - e^x}{x^2(e^x)^2} \\
&= \frac{(-x^3 + 6x^2 - x - 1)\cancel{e^x}}{x^2(e^x)^{\cancel{2}}} &&= \boxed{\frac{-x^3 + 6x^2 - x - 1}{x^2e^x}}
\end{aligned}$$

Notes: Use the quotient rule. Also, we need the product rule to help take the derivative of xe^x. The rest is algebra.

(d) $y = \Big(\ln(3x+1) + 55\Big)^{100}$

$$y' = 100\Big(\ln(3x+1) + 55\Big)^{99}\frac{1}{3x+1}(3) = \boxed{\frac{300\Big(\ln(3x+1) + 55\Big)^{99}}{3x+1}}$$

Notes: Use the chain rule (specifically the "generalized power rule") with outside function BLAH^{100} and inside function $\ln(3x+1) + 55$. To differentiate $\ln(3x+1)$ we need the chain rule again. This time $\ln(\text{BLAH})$ is our outside function and $3x + 1$ is our inside function.

(e) $y = \ln\left(\dfrac{4(x^5+10)^3 e^{2x}}{\sqrt{x}\,(x-8)^{10}}\right)$

First some algebra: $y = \ln\left(4(x^5+10)^3e^{2x}\right) - \ln\left(\sqrt{x}\,(x-8)^{10}\right) = \ln(4) + \ln(x^5+10)^3 + \ln(e^{2x}) - \ln\left(x^{1/2}\right) - \ln(x-8)^{10}$
$= \ln(4) + 3\ln(x^5+10) + \cancel{\ln(e}^{2x}) - \dfrac{1}{2}\ln(x) - 10\ln(x-8) = \ln(4) + 3\ln(x^5+10) + 2x - \dfrac{1}{2}\ln(x) - 10\ln(x-8)$

$$y' = 0 + 3\frac{1}{x^5+10}(5x^4) + 2 - \frac{1}{2}\cdot\frac{1}{x} - 10\,\frac{1}{x-8}(1) = \boxed{\frac{15x^4}{x^5+10} + 2 - \frac{1}{2x} - \frac{10}{x-8}}$$

Notes: The derivative of $\ln(4)$ is 0 since $\ln(4)$ is a constant (it has no x's in it!). $3\ln(x^5+10)$ can be differentiated using the chain rule. The same is true of $-10\ln(x-8)$.