Math 1030 Workshop #4 **Name:** ANSWER KEY

DUE: Friday, July 22$^{\text{nd}}$ Please turn in a paper copy and **SHOW YOUR WORK!**

1. Use the limit definition of the derivative to find $f'(x)$ if...
[You use should the rules we learned to double check your answer.]

(a) $f(x) = -4x^2 + 2x - 1$ then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$

$$\lim_{h\to 0} \frac{\left[-4(x+h)^2 + 2(x+h) - 1\right] - \left[-4x^2 + 2x - 1\right]}{h} = \lim_{h\to 0} \frac{\left[-4(x^2 + 2xh + h^2) + 2(x+h) - 1\right] - \left[-4x^2 + 2x - 1\right]}{h} =$$

$$\lim_{h\to 0} \frac{\left[-4x^2 - 8xh - 4h^2 + 2x + 2h - 1\right] - \left[-4x^2 + 2x - 1\right]}{h} = \lim_{h\to 0} \frac{-4x^2 - 8xh - 4h^2 + 2x + 2h - 1 + 4x^2 - 2x + 1}{h} =$$

$$\lim_{h\to 0} \frac{-8xh - 4h^2 + 2h}{h} = \lim_{h\to 0} \frac{h(-8x - 4h + 2)}{h} = \lim_{h\to 0} -8x - 4h + 2 = -8x - 4(0) + 2 = \boxed{-8x + 2}$$

[Of course, we can easily check this using linearity and the power rule: $f'(x) = -4(2x) + 2 = -8x + 2$.]

(b) $f(x) = \frac{1}{\sqrt{x}}$ then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} =$

$$\lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} \cdot \frac{\sqrt{x}}{\sqrt{x}} - \frac{1}{\sqrt{x}} \cdot \frac{\sqrt{x+h}}{\sqrt{x+h}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{x}}{\sqrt{x+h}\sqrt{x}} - \frac{\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h} = \lim_{h\to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}}{h} =$$

Let's pause to comment on the line above. We start with the limit definition applied to our function. Then in order to bring the fractions in the numerator together we force a common denominator (i.e. $\sqrt{x} \cdot \sqrt{x+h}$). Then with common denominator in hand we can combine these fractions.

$$\lim_{h\to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}}{\frac{h}{1}} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\sqrt{x}} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\,\sqrt{x} \cdot h} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h}\,\sqrt{x} \cdot h} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} =$$

In the above line we have performed a division of fractions. The last step is the beginning of what we call "the conjugate trick". This will allow (eventually) to cancel off the h in the denominator.

$$\lim_{h\to 0} \frac{(\sqrt{x} - \sqrt{x+h})(\sqrt{x} + \sqrt{x+h})}{\sqrt{x+h}\,\sqrt{x} \cdot h \cdot (\sqrt{x} + \sqrt{x+h})} = \lim_{h\to 0} \frac{(\sqrt{x})^2 - \sqrt{x}\sqrt{x+h} + \sqrt{x+h}\sqrt{x} - (\sqrt{x+h})^2}{\sqrt{x+h}\,\sqrt{x} \cdot h \cdot (\sqrt{x} + \sqrt{x+h})} =$$

$$\lim_{h\to 0} \frac{(\sqrt{x})^2 - (\sqrt{x+h})^2}{\sqrt{x+h}\,\sqrt{x} \cdot h \cdot (\sqrt{x} + \sqrt{x+h})} = \lim_{h\to 0} \frac{x - (x+h)}{\sqrt{x+h}\,\sqrt{x} \cdot h \cdot (\sqrt{x} + \sqrt{x+h})} = \lim_{h\to 0} \frac{x - x - h}{\sqrt{x+h}\,\sqrt{x} \cdot h \cdot (\sqrt{x} + \sqrt{x+h})} =$$

$$\lim_{h\to 0} \frac{-1 \cdot h}{\sqrt{x+h}\,\sqrt{x} \cdot h \cdot (\sqrt{x} + \sqrt{x+h})} = \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\,\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x+0}\,\sqrt{x}(\sqrt{x} + \sqrt{x+0})} =$$

In our next to last step we finally canceled off the h in the denominator. This means we can plug in $h = 0$ without having a division by zero issue. Now all that is left is to do some simplification.

$$\frac{-1}{\sqrt{x}\,\sqrt{x}(\sqrt{x} + \sqrt{x})} = \frac{-1}{x(2\sqrt{x})} = \frac{-1}{2x \cdot x^{1/2}} = \boxed{\frac{-1}{2x^{3/2}}}$$

[Double check the answer. $f(x) = 1/\sqrt{x} = 1/x^{1/2} = x^{-1/2}$. The power rule says that $f'(x) = -\frac{1}{2}x^{-3/2} = \frac{-1}{2x^{3/2}}$.]

2. Find the equation of the line tangent to the graph of $y = f(x)$ at $x = x_0$ if...

(a) $f(x) = x^3 - x^2 + x - 1$ and $x_0 = -2$

We need a point and a slope.

When $x = -2$, $f(x) = f(-2) = (-2)^3 - (-2)^2 + (-2) - 1 = -8 - 4 - 2 - 1 = -15$. So our tangent line passes through the point $(x, y) = (-2, -15)$.

The derivative of $f(x)$ is a formula for the slope of f's tangent lines. In particular, $f'(-2)$ is the slope of the tangent at $x = -2$. Using linearity and the power rule, we get $f'(x) = 3x^2 - 2x + 1$ so that $f'(-2) = 3(-2)^2 - 2(-2) + 1 = 17$.

Finally using point-slope we can find the equation of our tangent line: $y - (-15) = 17(x - (-2))$. Simplfying, we get: $y + 13 = 17x + 34$ and so $\boxed{y = 17x + 19}$.

Wolfram Alpha will confirm our answer: `tangent line x^3-x^2+x-1 at x=-2` ✓

(b) $f(x) = \ln(x)$ and $x_0 = 1$

Same as in part (a). $f(1) = \ln(1) = 0$ so our line passes through $(x, y) = (1, 0)$.

$f'(x) = \dfrac{1}{x}$ and so our slope is $f'(1) = \dfrac{1}{1} = 1$.

Using point-slope we get that $y - 0 = 1(x - 1)$ and so $\boxed{y = x - 1}$.

Wolfram Alpha will confirm our answer: `tangent line ln(x) at x=1` ✓

3. Compute the derivative of each of the following functions.

(a) $y = 8\sqrt[4]{x} - 7\ln(x) + 2e^x - \dfrac{9}{x^5} + 3x + \sqrt{57} \qquad = 8x^{1/4} - 7\ln(x) + 2e^x - 9x^{-5} + 3x + \sqrt{57}$

$$y' = 8 \cdot \frac{1}{4}x^{-3/4} - 7 \cdot \frac{1}{x} + 2e^x - 9(-5)x^{-6} + 3 + 0 = \boxed{2x^{-3/4} - \frac{7}{x} + 2e^x + 45x^{-6} + 3}$$

Wolfram Alpha will give a simplified answer: `derivative 8x^(1/4)-7 ln(x) +2e^x -9x^(-5)+3x+sqrt(57)` ✓

(b) $y = x^7 \ln(x + 1)$ Use the product rule: first part $= x^7$ and second part $= \ln(x + 1)$

Technically, we also need to use the chain rule when differentiating $\ln(x + 1) = \ln(\text{BLAH})$. It's derivative is $\dfrac{1}{\text{BLAH}} \cdot \text{BLAH}' = \dfrac{1}{x + 1} \cdot 1$.

$$y' = (x^7)' \ln(x + 1) + x^7(\ln(x + 1))' = \boxed{7x^6 \ln(x + 1) + x^7 \cdot \frac{1}{x + 1}}$$

Wolfram Alpha will give a simplified answer: `derivative x^7 ln(x+1)` ✓

(c) $y = \dfrac{x^2e^x + 5x - 1}{x^3 + 4}$ We need to use the quotient rule.

While using the quotient rule we'll also have to use the product rule to differentiate x^2e^x (part of the numerator).

$$y' = \frac{(x^2e^x + 5x - 1)'(x^3 + 4) - (x^2e^x + 5x - 1)(x^3 + 4)'}{(x^3 + 4)^2} = \boxed{\frac{(2xe^x + x^2e^x + 5)(x^3 + 4) - (x^2e^x + 5x - 1)(3x^2)}{(x^3 + 4)^2}}$$

Wolfram Alpha will give a simplified answer: `derivative (x^2e^x+5x-1)/(x^3+4)` ✓

(d) $y = \left(e^{x^3} + 11x - 6\right)^{45}$ We need to use the chain rule (i.e. generalized power rule).

We'll actually need to apply a second chain rule to the term e^{x^3} on the inside of the $(\text{BLAH})^{45}$. Recall that the derivative of e^{STUFF} is $e^{\text{STUFF}} \cdot \text{STUFF}'$, so the derivative of e^{x^3} is $e^{x^3}(3x^2)$.

$$y' = 45\left(e^{x^3} + 11x - 6\right)^{44}\left(e^{x^3} + 11x - 6\right)' = \boxed{45\left(e^{x^3} + 11x - 6\right)^{44}\left(e^{x^3}(3x^2) + 11\right)}$$

Wolfram Alpha will give a simplified answer: `derivative (e^(x^3)+11x-6)^45` ✓

(e) $y = \ln\left(\dfrac{11\sqrt[3]{x}\,e^{-8x}}{(x^2+1)^6(x-2)^{100}}\right)$ We must first use laws of logs to expand.

$$y = \ln\left(11x^{1/3}\,e^{-8x}\right) - \ln\left((x^2+1)^6(x-2)^{100}\right) = \ln(11) + \ln(x^{1/3}) + \ln(e^{-8x}) - \ln((x^2+1)^6) - \ln((x-2)^{100})$$

$$= \ln(11) + \frac{1}{3}\ln(x) - 8x - 6\ln(x^2+1) - 100\ln(x-2)$$

Now we can differentiate. Note that to differentiate $\ln(x^2+1)$ and $\ln(x-2)$ we'll have to use the chain rule. Which (for a natural log) looks like $(\ln(\text{BLAH}))' = \dfrac{1}{\text{BLAH}} \cdot \text{BLAH}'$.

$$y' = 0 + \frac{1}{3}\cdot\frac{1}{x} - 8 - 6\frac{1}{x^2+1}(2x) - 100\frac{1}{x-2}\cdot 1 = \boxed{\frac{1}{3x} - 8 - \frac{12x}{x^2+1} - \frac{100}{x-2}}$$

Wolfram Alpha will give a simplified answer: `derivative ln((11x^(1/3)e^(-8x))/((x^2+1)^6(x-2)^100))` ✓