Math 1110 **Some Theorems & Definitions** **Supplement**

Theorem: (Intermediate Value Theorem) Let $f(x)$ be continuous on the closed interval $[a, b]$. If y_0 is any real number between $f(a)$ and $f(b)$, then there exists some $a \leq x_0 \leq b$ such that $f(x_0) = y_0$. In other words, if $f(x)$ is continuous and hits two outputs, then it must hit all of the outputs in between as well.

Corollary: Let $f(x)$ be continuous on the closed interval $[a, b]$. If $f(a)$ and $f(b)$ have different signs (i.e. $f(a) < 0$ and $f(b) > 0$ or $f(a) > 0$ and $f(b) < 0$), then there exists some real number c such that $a < c < b$ and $f(c) = 0$. In other words, if $f(x)$ is continuous, it cannot switch from being positive to negative without hitting 0 (i.e. the x-axis) at some point.

Definition: If $f(x_1) < f(x_2)$ for all $a < x_1 < x_2 < b$, then $f(x)$ is **strictly increasing** on the open interval (a, b). If $f(x_1) \leq f(x_2)$ for all $a < x_1 < x_2 < b$, then $f(x)$ is **increasing** on (a, b). Likewise, if $f(x_1) > f(x_2)$ for all $a < x_1 < x_2 < b$, then $f(x)$ is **strictly decreasing** on (a, b). If $f(x_1) \geq f(x_2)$ for all $a < x_1 < x_2 < b$, then $f(x)$ is **decreasing** on (a, b).

Definition: If $f'(a) = 0$ or $f'(a)$ does not exist, then $x = a$ (or more precisely $(x, y) = (a, f(a))$) is a **critical point**.

Theorem: (Mean Value Theorem) Let $f(x)$ be continuous on the closed interval $[a, b]$ and differentiable (i.e. its derivative exists) on the open interval (a, b). Then there exists some c such that $a < c < b$ and

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

In other words, the derivative must take on the average value at some point.

Corollary: Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) > 0$ when $a < x < b$, then $f(x)$ is strictly increasing on the open interval (a, b). If $f'(x) \geq 0$ when $a < x < b$, then $f(x)$ is increasing on (a, b). Likewise, if $f'(x) < 0$ when $a < x < b$, then $f(x)$ is strictly decreasing on (a, b). If $f'(x) \leq 0$ when $a < x < b$, then $f(x)$ is decreasing on (a, b).

Note: If $f(x)$ has a *continuous* derivative, then $f'(a) > 0$ guarantees that $f'(x) > 0$ for x's "near" a, this means that there is some $\epsilon > 0$ such that $f'(x) > 0$ for all $a - \epsilon < x < a + \epsilon$. Thus by the above corollary, $f(x)$ is increasing on $(a - \epsilon, a + \epsilon)$. In the presence of a *continuous* derivative, having a positive derivative at $x = a$ is enough to guarantee that $f(x)$ is increasing on an interval centered about $x = a$. A similar statement holds for $f'(a) < 0$. That said – if $f(x)$ has a discontinuous derivative, it is *possible* to have a positive derivative at $x = a$, yet $f(x)$ "jiggles" up and down arbitrarily close to $x = a$ (so it does not increase on any interval surrounding $x = a$).

Theorem: If $f(x)$ is increasing (or strictly increasing) on $[a, b]$ and differentiable on (a, b), then $f'(x) \geq 0$. Likewise, if $f(x)$ is decreasing, then $f'(x) \leq 0$.

Note: Generally, if $f(x)$ is strictly increasing, we will have $f'(x) > 0$. But not always. For example, $f(x) = x^3$ is strictly increasing everywhere, but $f'(x) = 3x^2 = 0$ when $x = 0$. If $f'(a) = 0$, then $f(x)$ has a horizontal tangent at $x = a$ (i.e. $f(x)$ has temporarily "leveled off" at $x = a$).

Definition: $f(x)$ is **concave up** on (a, b) if given any $a < x_1 < x_2 < b$, then $f(x)$ lies below the secant line through $(x_1, f(x_1))$ and $(x_2, f(x_2))$. This is equivalent to saying $\dfrac{f(x) - f(x_1)}{x - x_1} \leq \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ for all $x_1 < x < x_2$. If $f(x)$ lies above its secant lines (so flip the "$\leq$" to "$\geq$" above), then it is **concave down**.

Theorem: If $f''(x) > 0$ for all $a < x < b$, then $f(x)$ is concave up on the open interval (a, b). Likewise, if $f''(x) < 0$ for all $a < x < b$, then $f(x)$ is concave down on (a, b).

Note: The same discussion holds for concavity as it did for increasing vs. decreasing. If $f(x)$ has a *continuous* second derivative, then $f''(a) > 0$ means $f(x)$ is concave up near $x = a$ and if $f''(a) < 0$, then $f(x)$ is concave down near $x = a$. But if $f''(x)$ is discontinuous, weird things can happen.

Theorem: Assume $f''(x)$ exists on an open interval (a, b). If $f(x)$ is concave up on (a, b), then $f''(x) \geq 0$ for all $a < x < b$. Likewise, if $f(x)$ is concave down on (a, b), then $f''(x) \leq 0$ for all $a < x < b$.

Note: Again, generally concave up will mean $f''(x) > 0$ and concave down will mean $f''(x) < 0$, but we can have $f''(x) = 0$ at points in regions where $f(x)$ is concave down or up. If $f''(a) = 0$, $f(x)$ should look linear near $x = a$ (i.e. $f(x)$ should look "flat" near $x = a$).

In summary, *roughly*, $f'(x) > 0$ means $f(x)$ is increasing, $f'(x) < 0$ means $f(x)$ is decreasing, $f'(x) = 0$ means $f(x)$ has leveled off. $f''(x) > 0$ means $f(x)$ is concave up, $f''(x) < 0$ means $f(x)$ is concave down, and $f''(x) = 0$ means $f(x)$ is flat.

Definition: If $f(x)$ switches from concave up to concave down at $x = a$, then $x = a$ (or more precisely $(x, y) = (a, f(a))$) is an **inflection point** of $f(x)$.

Definition: If $f(a) \geq f(x)$ for all x's in the domain of $f(x)$, then $x = a$ (or more precisely $(x, y) = (a, f(a))$) is a **global maximum** of $f(x)$. In this case, we call $f(a)$ the **maximum value** of $f(x)$ [think: value=output]. Likewise, if $f(a) \leq f(x)$ for all x's, then $x = a$ is a **global minimum** of $f(x)$ and $f(a)$ is $f(x)$'s **minimum value**.

Definition: If there exists some $\epsilon > 0$ such that $f(a) \geq f(x)$ for all $a - \epsilon < x < a + \epsilon$, then $x = a$ (or more precisely $(x, y) = (a, f(a))$) is a **local** or **relative maximum** of $f(x)$. Likewise, if there exists some $\epsilon > 0$ such that $f(a) \leq f(x)$ for all $a - \epsilon < x < a + \epsilon$, then $x = a$ (or more precisely $(x, y) = (a, f(a))$) is a **local** or **relative minimum** of $f(x)$. Roughly, $x = a$ is a local max if $f(x) \leq f(a)$ for all x's close to a, and $x = a$ is a local min if $f(x) \geq f(a)$ for all x's close to a.

Note: We use the term *extreme* to mean either a maximum or minimum. For example, the set of local extrema is the set of all local minimums and local maximums. A function's *extreme values* are its maximum and minimum values. By the way, a function can have many local minima and maxma. In fact, a function can have many global minima and maxima. For example, $f(x) = \sin(x)$ hits its maximum value of 1 and minimum value of -1 infinitely many times! On the other hand, a function can have at most one maximum *value* and at most one minimum *value*.

Warning: Endpoints are weird. Some textbooks call endpoints critical points and consider them local extrema, other books do not. For this class, I will leave their status ambiguous. However, if endpoints are allowed to be extrema, then they also need to be allowed as critical points.

Theorem: If $x = a$ is a local extreme of $f(x)$, then $x = a$ is a critical point of $f(x)$.

Note: The converse of this theorem is not true. For example, $x = 0$ is a critical point for $f(x) = x^3$ (since $f'(x) = 3x^2$ and so $f'(0) = 0$). However, $x = 0$ is neither a local min nor a local max of $f(x) = x^3$. So keep in mind that some of our critical points are likely to be min's or max's, but some of them may be neither. On the other hand, any min or max *must* be a critical point.

Theorem: If $x = a$ is an inflection point and $f''(x)$ is continuous near a, then $f''(a) = 0$.

Note: Again, the converse is certainly not true. For example, $f(x) = x^4$ is concave up everywhere (so $f(x)$ has no inflection points), but $f''(x) = 12x^2$ is 0 at $x = 0$. As before, if $f''(a) = 0$, then $x = a$ *might* be an inflection point but it does not have to be one. On the other hand, if $x = a$ is an inflection point and $f''(x)$ exists near $x = a$, then $f''(a)$ *must* be 0.

Theorem: (Extreme Value Theorem) Let $f(x)$ be continuous on the closed interval $[a, b]$. Then $f(x)$ attains a maximum and minimum value on $[a, b]$. In particular, there exists some $a \leq m \leq b$ and some $a \leq M \leq b$ such that $f(m) \leq f(x) \leq f(M)$ for all $a \leq x \leq b$ (i.e. $f(m)$ is the minimum value and $f(M)$ is the maximum value of $f(x)$ on $[a, b]$).

Note: If $f(x)$ is not continuous or if we are not working on a *closed* interval, then there is no guarantee that $f(x)$ has a global min or max.

Algorithm: If $f(x)$ is continuous and we wish to find the extreme values of $f(x)$ defined on $[a, b]$, then by the Extreme Value Theorem we know that these extreme values must exist. The global extremes must be found among the local extremes (or at the endpoints). Thus we find all of the local extrema and then decide which are global extrema. In particular...

- Differentiate $f(x)$.
- Find all of the critical points – that is – solve $f'(x) = 0$ for x and determine where $f'(x)$ does not exist.
- Throw away any critical point falling outside of the relevant interval $[a, b]$ (if a critical point is smaller than a or larger than b, it should be discarded).
- Plug each of the *relevant* critical points as well as the endpoints ($x = a$ and $x = b$) into the original function $f(x)$.
- The largest value of $f(x)$ in our list is the maximum value and the smallest (=most negative) is the minimum value.

Currently Relevant Problems from Sections 4.1 and 4.2:

Section 4.1: 5, 7, 9, 25, 26, 27, 43, 45

Section 4.2: 5, 7, 9, 11, 41, 44, 45