Math 1110	Taylor Polynomials	Supplement

Approximation is a major theme of calculus. In particular, we seek to understand a non-linear function in terms of a flattened out *linearized* function. If we linearize our function at a point, then the graph of its linearization is called a tangent line (based at that point) and the slope of the tangent line is the derivative of our function.

Sometimes a linear approximation is not good enough for our purposes. Thus we should develop higher order approximations. A first order approximation is called a linearization and a second order approximation is called a *quadratic approximation*. More generally these as well as higher order approximations are called *Taylor polynomials*. In second semester calculus these are extended to infinite sums called series. In turns out that in many cases Taylor *series* can be used to *define* our most familiar functions.

We begin by recalling the definition of a linearization and tangent.

Definition: Suppose $f(x)$ is differentiable at $x = a$ (i.e., $f'(a)$ exists). Then $L(x) = f(a) + f'(a)(x - a)$ is the **linearization** of $f(x)$ based (or centered) at $x = a$. The graph of this linearization is the **tangent line** of $f(x)$ based at $x = a$.

Example: Let $f(x) = x^2$. Then $f'(x) = 2x$. This means $f(3) = 9$ and $f'(3) = 6$ so that the linearization of $f(x) = x^2$ based at $x = 3$ is $L(x) = f(3) + f'(3)(x - 3) = 9 + 6(x - 3) = 6x - 9$. Thus the graph of $y = 6x - 9$ gives us the tangent line of $f(x) = x^2$ at $x = 3$.

Example: Let $g(x) = \sin(x)$. Then since $g'(x) = \cos(x)$, we have $L(x) = \sin(0) + \cos(0)(x - 0) = 0 + 1(x - 0) = x$ is the linearization of the sine function based at $x = 0$. This linear approximation is often used by Physicists when an angle is "small". In particular, $\sin(0.1) = g(0.1) \approx L(0.1) = 0.1$.

The notion of a "good" linearization can be turned around to define what we mean by *differentiability*. Say $L(x) = A + B(x - a)$. Then we might say $L(x)$ is a good linear approximation of $f(x)$ if $f(x) - L(x)$ is small (in some sense) when x is close to a. More precisely we want $\displaystyle\lim_{x\to a} \frac{f(x) - L(x)}{x - a} = 0$. In other words, we want $f(x)$ and $L(x)$ to match up to first order terms *locally*. If this definition is pursued, one can show that a good approximation exists only if $f'(a)$ exists and we have $A = f(a)$ and $B = f'(a)$. Briefly, we only have a "good" approximation at $x = a$ when $f(x)$ is differentiable and this good approximation must be the linearization $L(x) = f(a) + f'(a)(x - a)$.

Now we turn to a higher order approximation. Notice that if $L(x) = f(a) + f'(a)(x - a)$, then $L(a) = f(a)$ and $L'(a) = f'(a)$. Suppose that we want an approximation, say $Q(x)$, such that $Q(a) = f(a)$, $Q'(a) = f'(a)$, and $Q''(a) = f''(a)$. If we demand that $Q''(x)$ is the number $f''(a)$, then $Q'(x)$ would need to be linear: $Q'(x) = ?? + f''(a)x$ which could be rewritten as $Q'(x) = ??? + f''(a)(x - a)$ but if we want $Q'(a) = f'(a)$ we must have $Q'(x) = f'(a) + f''(a)(x - a)$. If we know about antiderivatives, we can conclude that $Q(x) = ??? + f'(a)(x - a) + \dfrac{f''(a)}{2}(x - a)^2$ and since we want $Q(a) = f(a)$ we must have $Q(x) = f(a) + f'(a)(x - a) + \dfrac{f''(a)}{2}(x - a)^2$.

Definition: Suppose $f(x)$ is twice differentiable at $x = a$. Then $Q(x) = f(a) + f'(a)(x - a) + \dfrac{f''(a)}{2}(x - a)^2$ is the **quadratic approximation** of $f(x)$ based (or centered) at $x = a$.

From our discussion above, this is the unique quadratic polynomial which matches $f(x)$ and its first two derivatives at the point $x = a$.

Example: Let $f(x) = x^3$. Then $f'(x) = 3x^2$ and $f''(x) = 6x$. The quadratic approximation of $f(x) = x^3$ centered at $x = 1$ is $Q(x) = f(1) + f'(1)(x - 1) + \dfrac{f''(1)}{2}(x - 1)^2 = 1 + 3(x - 1) + 3(x - 1)^2$.

Example: Let $g(x) = \cos(x)$. Then $g'(x) = -\sin(x)$ and $g''(x) = -\cos(x)$. So $Q(x) = \cos(0) - \sin(0)x + \dfrac{-\cos(0)}{2}x^2 = 1 - \dfrac{1}{2}x^2$ is the quadratic approximation of cosine based at $x = 0$. Since $x = 0$ is a critical point for cosine, we have no linear term. Notice that the graph of this quadratic approximation is a parabola opening downward. This reveals that $x = 0$ is a local maximum for cosine.

Extending the above example gives us the *second derivative test*. Suppose that $f(x)$ has a second derivative at $x = a$ and that $x = a$ is a critical point (so $f'(a) = 0$). Then $f(x) \approx Q(x) = f(a) + f'(a)(x - a) + \dfrac{f''(a)}{2}(x - a)^2 = f(a) + \dfrac{f''(a)}{2}(x - a)^2$. The graph of $Q(x)$ is a parabola with vertex at $(a, f(a))$. It opens up or down according to the sign of $f''(a)$. Thus since $f(x) \approx Q(x)$ near $x = a$, it is reasonable to conclude that $x = a$ is a local minimum if $f''(a) > 0$ and a local maximum if $f''(a) < 0$.

We now generalize to arbitrary order approximations. First, recall that $k! = k(k-1)(k-2)\cdots 3\cdot 2\cdot 1$ is the factorial function. For example, $1! = 1$, $2! = 2$, $3! = 3\cdot 2\cdot 1 = 6$, $4! = 4\cdot 3\cdot 2\cdot 1 = 24$ etc. Since an empty product should be the multiplicative identity, one defines $0! = 1$ (admittedly this seems strange at first glance).

Definition: Let $f(x)$ have derivatives up to order n defined at $x = a$. Then the n^{th}*-order Taylor polynomial centered at* $x = a$ of $f(x)$ is

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x-a)^k$$

Taylor polynomials centered at $x = 0$ are traditionally called *Maclaurin polynomials*. Notice that a linearization is just a first order Taylor polynomial and a quadratic approximation is just a second order Taylor polynomial.

Example: Let $f(x) = \ln(x)$. Then $f'(x) = 1/x = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, and $f^{(4)}(x) = -6x^{-4}$. Thus $f(1) = \ln(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, and $f^{(4)}(1) = -6$. This means that the fourth order Taylor polynomial for the natural logarithm (centered at $x = 1$) is:

$$P(x) = 0 + 1(x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}.$$

Theorem: If $P(x)$ is the n^{th}-order Taylor polynomial of $f(x)$ centered at $x = a$, then we have $f(a) = P(a)$, $f'(a) = P'(a)$, $\ldots$, $f^{(n)}(a) = P^{(n)}(a)$. Conversely, suppose $P(x)$ is a polynomial of order n or less and $f(a) = P(a)$, $f'(a) = P'(a)$, $\ldots$, $f^{(n)}(a) = P^{(n)}(a)$. Then $P(x)$ is the n^{th}-order Taylor polynomial of $f(x)$ centered at $x = a$.

The proof of this theorem is straight forward (but we skip it). It tells us that the Taylor polynomial centered at $x = a$ is *only way* to approximate $f(x)$ with a polynomial of order (at most) n so that this polynomial's derivative data at $x = a$ matches $f(x)$ (up to order n).

Example: Suppose $P(x) = 5 + 4(x+8)^2 - (x+8)^3$ is the third order Taylor polynomial centered at $x = -8$ for some mystery function $f(x)$. Then we must have $f(-8) = P(-8) = 5$, $f'(-8) = P'(-8) = 0$, $f''(-8) = P''(-8) = 4\cdot 2 = 8$, and $f'''(-8) = P'''(-8) = (-1)\cdot 3! = -6$. And that is all we can say about f's values! We can draw no conclusions about $f^{(4)}(-8)$ or $f(2)$ or $f'(1)$ etc.

Example: Find the fourth order Taylor polynomial, centered at $x = 2$ for $g(x) = x^4$. This is easy. Answer: $P(x) = x^4$. Why? Well, $g(x)$ *is* a fourth order polynomial and it matches all of the derivative data it needs to, so by the above theorem it must be the Taylor polynomial! What is the 100-th order Taylor polynomial of $g(x) = x^4$ centered at $x = \sqrt{\pi}$? Easy, it's just $g(x) = x^4$.

Likewise, if $h(x) = x^2 - 3x + 1$. Then the zero-th order Maclaurin polynomial for $h(x)$ is 1, the first order Maclaurin polynomial is $-3x + 1$, and the Macluarin polynomials of degree two or higher are just $h(x) = x^2 - 3x + 1$ itself!

Sometimes we need to know exactly how good our estimate is. The following theorem does just that and is called *Taylor's Remainder Formula*.

Theorem: Suppose $f(x)$ is $(n+1)$-times differentiable on an interval containing a. Then for any x in that interval there is some number between x and a, say ζ_x, such that:

$$f(x) - P(x) = f(x) - \left(f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \right) = \frac{f^{(n+1)}(\zeta_x)}{(n+1)!}(x-a)^{n+1}$$

where $P(x)$ is the n-th order Taylor polynomial of $f(x)$ based at $x = a$.

Possibly surprisingly, this theorem is not terribly hard to prove (yet we still skip the proof here). It follows quickly from Rolle's theorem. What this tells us is that $f(x)$ and $P(x)$ are nearly identical. We just need to add in one more term – except that term's derivative may need to be evaluated at ζ_x (possibly near but not quite at a).

This formula says if we know how big the $(n+1)$-st derivative can be, we can get an estimate of how far off our Taylor polynomial is relative to how far we have strayed from our base point $x = a$.

Example: The fourth order Maclaurin polynomial for sine is $P(x) = x - \dfrac{x^3}{6}$. Note that the fifth derivative of sine is cosine. So Taylor says $\sin(x) - P(x) = \dfrac{\cos(\zeta_x)}{5!}(x-0)^5 = \dfrac{\cos(\zeta_x)}{120}x^5$ for some ζ_x between x and 0. But cosine is at most 1. Thus $|\sin(x) - P(x)| \leq \dfrac{1}{120}|x|^5$. For example, $\sin(0.1) \approx P(0.1) = 0.0998333\cdots$. How good is this approximation? Taylor says its error is less than $\dfrac{(0.1)^5}{120} \approx 8.333\cdots \times 10^{-8}$. So at least 6 decimals must match. In fact, $\sin(0.1) = 0.0998334166\cdots$. Thus this cubic polynomial is approximating sine quite well! (At least for small x's.)

Example: Approximate e (Euler's number) out to 3 decimals. We have $P(x) = 1 + x + \dfrac{x^2}{2} + \cdots + \dfrac{x^n}{n!}$ is the n-th order Maclaurin polynomial for $f(x) = e^x$. Suppose we at least know $e < 3$. Then $f^{(n+1)}(\zeta_1) = e^{\zeta_1} \leq e^1 < 3$

for $0 \leq \zeta_1 \leq 1$. Thus if we wish to approximate $f(1) = e^1 = e$ with $P(1)$, Taylor says our error is no more than $\dfrac{f^{(n+1)}(\zeta_1)}{(n+1)!}(1-0)^{n+1} < \dfrac{3}{(n+1)!}$. We want the error to be smaller than 10^{-3} so our answer is correct out to 3 digits. We need $\dfrac{3}{(n+1)!} < 10^{-3}$ so that $3 \times 10^3 = 3000 < (n+1)!$. Since $7! = 5040$, $n = 6$ should work for us. So $e \approx 1 + 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{6!} = 1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} = 2.7180555\cdots$. *Note:* $e = 2.7182818\cdots$ so our approximation is correct to 3 digits exactly!

Taylor polynomials also give us a way to extend our second derivative test. Recall that the second derivative test does not apply when the second derivative is zero at a critical point. We now offer a test that overcomes this problem.

Theorem: (n-th Derivative Test) Let $f(x)$ be a function such that $f^{(n)}(a)$ exists and $f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0$ (i.e., the first $n-1$ derivatives of f vanish at $x = a$) but $f^{(n)}(a) \neq 0$. We have the following:

- If n is odd, then $x = a$ is not a local extreme (it is neither a local minimum nor a local maximum).
- If n is even and $f^{(n)}(a) > 0$, then $x = a$ is a local minimum.
- If n is even and $f^{(n)}(a) < 0$, then $x = a$ is a local maximum.

Proof idea: The n-th order Taylor polynomial of $f(x)$ based at $x = a$ is $P(x) = f(a) + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ (since all lower order derivatives are zero). Near $x = a$, higher powers of $x - a$ are much closer to zero than lower powers, so $(x-a)^n$ dominates $(x-a)^{n+1}$. Thus the Taylor remainder (a multiple of $(x-a)^{n+1}$) is small enough that it does not matter. Therefore, the graphs of $f(x)$ and $P(x) = f(a) + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ are almost the same near $x = a$. Notice that $x - a$, $(x-a)^3$, $(x-a)^5$, etc. do not have an extrema at $x = a$ while $(x-a)^2$, $(x-a)^4$, etc. do. Thus if n is odd, $x = a$ is not an extremum. If n is even, $x = a$ is a local extreme. Moreover, this extrema is a minimum or maximum according to whether the coefficient of $(x-a)^n$ is positive or negative.

Example: $f(x) = (x-3)^4$ has a critical point at $x = 3$. Notice that $f'(x) = 4(x-3)^3$, $f''(x) = 12(x-2)^2$, and $f'''(x) = 24(x-2)$ are zero at $x = 3$. However, $f^{(4)}(x) = 24$ so $f^{(4)}(3) = 24 > 0$ (our first non-zero derivative has even order and is a positive number). Thus $x = 3$ is a local minimum. On the other hand, $f'(10) = 4(10-3)^3 \neq 0$. Thus since our first non-zero derivative at $x = 10$ is of odd order (i.e., order 1), our test says $x = 10$ is neither a minimum nor a maximum. This really is not a surprise since $x = 10$ is not even a critical point!

Homework:

1. Compute the quadratic approximation of $f(x) = e^x$ based at $x = 0$ (i.e., the second order Maclaurin polynomial).
2. Find the third order Taylor polynomial of $f(x) = \ln(x-2)$ based at $x = 3$.
3. Find the third order Maclaurin polynomial of $f(x) = \arctan(x)$.
4. Find Taylor polynomials based at $x = -1$ of $f(x) = x^3$ of all possible orders.
5. Compute the fifth order Maclaurin polynomials for sine, cosine, and e^x.
6. Suppose $f(x) = x^4 + 3x^2 - x + 1$. What is the 10-th order Taylor polynomial of $f(x)$ based at $x = -7$?
7. Suppose $P(x) = 5 - 4(x+1) + 2(x+1)^3 - (x+1)^4$ is the fifth order Taylor polynomial based at $x = -1$ for $f(x)$. Find $f(-1)$, $f'(-1)$, $f''(-1)$, $f'''(-1)$, and $f^{(4)}(-1)$.
8. Suppose $P(x) = x^3$ is the fourth order Taylor polynomial based at $x = 2$ for $f(x)$. Find $f(2)$, $f'(2)$, $f''(2)$, $f'''(2)$, and $f^{(4)}(2)$. What about $f^{(5)}(2)$? How about $f'(3)$?
9. Use Taylor's remainder formula to explain why if $P(x)$ is a Taylor polynomial of order k where $k \geq n$ based at $x = a$ for a polynomial $f(x)$ of order n, then $P(x) = f(x)$.
10. Approximate $\cos(0.1)$ using a fourth order Maclaurin polynomial. Use Taylor's remainder formula to estimate the possible error.

$\boxed{\textbf{Answers:}}$

1. All derivatives of $f(x) = e^x$ are just itself. Thus $f^{(k)}(0) = e^0 = 1$. So $Q(x) = 1 + 1 \cdot x + \frac{1}{2} \cdot x^2 = \frac{1}{2}x^2 + x + 1$.

2. We have $f'(x) = \frac{1}{x-2} = (x-2)^{-1}$, $f''(x) = -(x-2)^{-2}$, and $f'''(x) = 2(x-2)^{-3}$. Thus $f(3) = \ln(3-2) = \ln(1) = 0$, $f'(3) = (3-2)^{-1} = 1$, $f''(3) = -(3-2)^{-2} = -1$, and $f'''(3) = 2(3-2)^{-3} = 2$. Therefore, $P(x) = 0 + 1(x-3) + \frac{-1}{2}(x-3)^2 + \frac{2}{3!}(x-3)^3 = (x-3) - \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3$.
   One might (correctly) guess that the n-th order Taylor polynomial based at $x = 3$ would be: $P(x) = (x-3) - \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3 - \frac{1}{4}(x-3)^4 + \cdots \pm \frac{1}{n}(x-3)^n$.

3. Recall that a Maclaurin polynomial is just a Taylor polynomial based at $x = 0$. Taking derivatives we get: $f'(x) = \frac{1}{x^2+1}$, $f''(x) = \frac{-2x}{(x^2+1)^2}$, and $f'''(x) = \frac{-2(x^2+1)^2 + 2x(2)(x^2+1)2x}{(x^2+1)^4}$ so that $f(0) = \arctan(0) = 0$, $f'(0) = \frac{1}{0^2+1} = 1$, $f''(x) = \frac{-2(0)}{(0^2+1)^2}$, and $f'''(0) = \frac{-2(0^2+1)^2 + 0}{(0^2+1)^4} = -2$.
   Therefore, $P(x) = 0 + 1 \cdot x + \frac{0}{2} \cdot x^2 + \frac{-2}{3!} \cdot x^3 = x - \frac{1}{3}x^3$.

4. Computing derivatives: $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$, and $f^{(k)}(x) = 0$ for $k > 3$ so that $f(-1) = -1$, $f'(-1) = 3$, $f''(-1) = -6$, $f'''(-1) = 6$, and $f^{(k)}(-1) = 0$ for $k > 3$. Therefore, the zero-th order polynomial is $P(x) = -1$, first order is $P(x) = -1 + 3(x-1)$, second order is $P(x) = -1 + 3(x-1) + \frac{-6}{2}(x-1)^2 = -1 + 3(x-1) - 3(x-1)^2$, and third order is $P(x) = -1 + 3(x-1) + \frac{-6}{2}(x-1)^2 + \frac{6}{3!}(x-1)^3 = -1 + 3(x-1) - 3(x-1)^2 + (x-1)^3$ which is actually equal to x^3 itself! Higher order terms contribute nothing since $f^{(k)}(-1) = 0$ for $k > 3$. In fact, all Taylor polynomials of order 3 or more based at any point are just $f(x) = x^3$ itself!

5. Derivatives of $\sin(x)$ are $\cos(x)$, $-\sin(x)$, $-\cos(x)$, $\sin(x)$ and so on. Evaluating $\sin(x)$ itself and these derivatives at $x = 0$ yields $0, 1, 0, -1, 0$, etc. The same data is needed for cosine. We have that the fifth order Maclaurin polynomial for $\sin(x)$ is $P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$. Likewise, the fifth order Maclaurin polynomial for $\cos(x)$ is $P(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$.
   Recall from problem #1, the derivatives of e^x are just itself and so at $x = 0$ we just get $e^0 = 1$. Thus the fifth order Maclaurin polynomial of e^x is $P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$.
   It should be easy to guess what higher order Maclaurin polynomials for sine, cosine, and e^x are.

6. Any Taylor polynomial of order 4 or more based at any point is just going to be $f(x) = x^4 + 3x^2 - x + 1$ itself.

7. In a Taylor polynomial of $f(x)$ based at $x = -1$, the coefficient of $(x+1)^k$ is $\frac{f^{(k)}(-1)}{k!}$. So if the coefficient of $(x+1)^k$ is c, we have $f^{(k)}(-1) = c \cdot k!$. Thus $f(-1) = 5$, $f'(-1) = -4$, $f''(-1) = 0 \cdot 2 = 0$ (there is no $(x+1)^2$ term), $f'''(-1) = 2 \cdot 3! = 12$, and $f^{(4)}(-1) = -1 \cdot 4! = -24$.

8. First, note that $P'(x) = 3x^2$, $P''(x) = 6x$, $P'''(x) = 6$, and $P^{(4)}(x) = 0$. Since a function and its fourth order Taylor polynomial based at $x = 2$ must have matching data at $x = 2$ (up to the fourth derivative), we get that $f(2) = P(2) = 2^3 = 8$, $f'(2) = P'(2) = 3 \cdot 2^2 = 12$, $f''(2) = P''(2) = 6 \cdot 2 = 12$, $f'''(2) = P'''(2) = 6$, and $f^{(4)}(2) = P^{(4)}(2) = 0$.
   This is as much as we can say. While $P^{(5)}(2) = 0$, there is no guarantee that $f^{(5)}(2)$ matches this value. In the end, $f^{(5)}(2)$ is completely unknowable from the data given. Likewise, this Taylor polynomial does not tell us about data at other inputs, so $f'(3)$ is also unknowable from our given information.

9. Notice that when $f(x)$ is a polynomial of order n, $f^{(k)}(x) = 0$ for $k > n$. Thus the Taylor remainder formula is zero for polynomials of order n or more.

10. Using work from problem #5 to get the fourth order Maclaurin polynomial for cosine, we have $\cos(0.1) \approx 1 - \frac{0.1^2}{2} + \frac{0.1^4}{4!} = 0.9550041\overline{666}$. Since the fifth derivative of cosine is $-\sin(x)$ and $|-\sin(x)| \leq 1$, the Taylor remainder formula caps the error at $\frac{1}{5!}(0.1)^5 = 0.00000008\overline{333}$. *Note:* $\cos(0.1) = 0.995004165278\cdots$.