1. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n n^2}$ Find the radius and interval of convergence.

To find the radius of convergence we need to apply the (generalized) ratio test.

$$\lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{3^{n+1}(n+1)^2}}{\frac{(x-2)^n}{3^n n^2}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1} 3^n n^2}{(x-2)^n 3^{n+1} (n+1)^2} \right| = \lim_{n\to\infty} |x-2| \frac{n^2}{3(n+1)^2} = \frac{|x-2|}{3}$$

So the series converges if $|x-2| < 3$ and diverges if $|x-2| > 3$ (the radius of convegence is $R = 3$). Let's see what happens at the endpoints.

Left endpoint $x = x_0 - R = 2 - 3 = -1$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1-2)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which **converges** since the absolute value of its terms gives a p-series with $p = 2 > 1$.

Right endpoint $x = x_0 + R = 2 + 3 = 5$: $\displaystyle\sum_{n=1}^{\infty} \frac{(5-2)^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{3^n}{3^n n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which **converges** since it's a p-series with $p = 2 > 1$.

Answer: The radius of convergence is $R = 3$ and the interval of convergence is $I = [-1, 5]$ (both endpoints are included).

2. Find the power series expansion (centered at $x_0 = 0$) for $f(x) = \dfrac{1}{(1-2x)^2}$.

We know that $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ (the geometric series). Differentiating we get

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[(1-x)^{-1}\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} x^n\right] = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{\ell=0}^{\infty} (\ell+1) x^{\ell}$$

Finally, plug-in $2x$ for x and get $\displaystyle\frac{1}{(1-2x)^2} = \sum_{\ell=0}^{\infty} (\ell+1)(2x)^{\ell}$.

By the way, the series for $(1-x)^{-2}$ converges when $|x| < 1$, so our new series converges when $|2x| < 1$ which is $|x| < 1/2$.

Answer: $f(x) = \displaystyle\frac{1}{(1-2x)^2} = \sum_{\ell=0}^{\infty} (\ell+1)(2x)^{\ell}$ (when $|x| < 1/2$)

3. Consider the power series $\sum_{n=0}^{\infty} a_n(x+1)^n$. We know that $\sum_{n=0}^{\infty} a_n 2^n$ converges and $\sum_{n=0}^{\infty} a_n(-5)^n$ diverges. What can we say about the radius of convergence of our power series?

Briefly, the convergence of $\sum_{n=0}^{\infty} a_n 2^n$ guarantees the convergence of $\sum_{n=0}^{\infty} a_n(x+1)^n$ as long as $|x+1| < 2$. On the other hand, the divergence of $\sum_{n=0}^{\infty} a_n(-5)^n$ guarantees the divergence of $\sum_{n=0}^{\infty} a_n(x+1)^n$ as long as $|x+1| > |-5| = 5$. Therefore, if R is the radius of convergence, then $2 \leq R \leq 5$.

Alternatively, we could note that $2 = 1 - (-1)$ and $-5 = -6 - (-1)$. So we are given convergence when $x = 1$ and divergence when $x = -6$. Since $x = 1$ is distance 2 from $x_0 = -1$, we are guaranteed a radius of convergence $R \geq 2$. Likewise, since $x = -6$ is distance 5 from $x_0 = -1$, we are gauranteed a radius of convergence $R \leq 5$.

Answer: $2 \leq R \leq 5$.