1. $\displaystyle\int_{-\infty}^{0} e^{3x}\,dx = \lim_{a\to-\infty}\int_a^0 e^{3x}\,dx = \lim_{a\to-\infty}\frac{1}{3}e^{3x}\Big|_a^0 = \lim_{a\to-\infty}\frac{1}{3}e^0 - \frac{1}{3}e^{3a} = \frac{1}{3} - 0 = \frac{1}{3}$
(Converges to 1/3)

2. $\displaystyle\int_0^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx = \lim_{b\to\infty} -xe^{-x}\big|_0^b + \int_0^b e^{-x}\,dx = \lim_{b\to\infty} -\frac{b}{e^b} + 0\,\left(-e^{-x}\right)\big|_0^b$
$\displaystyle = \lim_{b\to\infty} -\frac{b}{e^b} - \frac{1}{e^b} + e^0 = 1$ Using L'Hopital's rule to evaluate $\lim b/e^b$.
(Converges to 1)

3. $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{\ln(x)}}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\sqrt{\ln(x)}}\,dx = \lim_{b\to\infty}\int_{\ln(2)}^{\ln(b)} \frac{1}{\sqrt{u}}\,du$
Substitute $u = \ln(x)$ (so that $du = (1/x)dx$, $2 \mapsto \ln(2)$, and $b \mapsto \ln(b)$).
$\displaystyle = \lim_{b\to\infty}\int_{\ln(2)}^{\ln(b)} u^{-1/2}\,du = \lim_{b\to\infty}\frac{u^{1/2}}{1/2}\Big|_{\ln(2)}^{\ln(b)} = \lim_{b\to\infty} 2\sqrt{\ln(b)} - 2\sqrt{\ln(2)} = \infty$
(Diverges)

4. $\displaystyle\int_4^{\infty} \frac{2}{x^2-1}\,dx$
Note: $x^2 - 1 = 0$ only when $x = \pm 1$, so this integral is only improper at ∞. Also, we need to find the partial fraction decomposition before integrating.
$$\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$
Then after clearing the denominators, $2 = A(x+1) + B(x-1)$. Plugging in $x = -1$, we find that $2 = 0 + B(-2)$ so $B = -1$. Plugging in $x = 1$, we get that $2 = A(2) + 0$ so $A = 1$. Therefore,
$\displaystyle = \lim_{b\to\infty}\int_4^b \frac{1}{x-1} + \frac{-1}{x+1}\,dx = \lim_{b\to\infty} \ln|x-1| - \ln|x+1|\big|_4^b = \lim_{b\to\infty} \ln\left|\frac{x-1}{x+1}\right|\Big|_4^b$
In the last step we used the law of logs: $\ln(X) - \ln(Y) = \ln(X/Y)$.
$\displaystyle = \lim_{b\to\infty} \ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{4-1}{4+1}\right| = \ln(1) - \ln\left(\frac{3}{5}\right) = \ln\left(\frac{5}{3}\right)$
Note: $(b-1)/(b+1) \to 1$ as $b \to \infty$ so the first log expression approaches $\ln(1) = 0$.
In the last step we used the fact that $-1\ln(X) = \ln(X^{-1})$.
(Converges to $\ln(5/3)$)

5. $\displaystyle\int_0^{\infty} \frac{1}{x^2}\,dx$
WARNING: $1/x^2$ is improper at $x = 0$ (as well as $x = \infty$). So we need to split up this integral to deal with both endpoints separately. Let's split the integral at $x = 1$. That is... $\int_0^\infty = \int_0^1 + \int_1^\infty$.
$\displaystyle\int_0^1 \frac{1}{x^2}\,dx = \lim_{a\to0^+}\int_a^1 x^{-2}\,dx = \lim_{a\to0^+} -x^{-1}\big|_a^1 = \lim_{a\to0^+}\frac{-1}{1} - \frac{-1}{a} = \infty$
Since the $\int_0^1$-part diverges, whole integral diverges. [However, the $\int_1^\infty$-part does converge. Although this isn't important to our answer.]
(Diverges)

6. $\displaystyle\int_0^3 \frac{dx}{x-2}$
Note: this integral is improper when $x-2=0$ (i.e. when $x=2$). Since this integral is improper "in the middle" we need to split it up into 2 pieces. $\int_0^3 = \int_0^2 + \int_2^3$. Let's deal with the $\int_2^3$-part first.
$$\int_2^3 \frac{dx}{x-2} = \lim_{a\to 2^+} \int_a^3 \frac{dx}{x-2} = \lim_{a\to 2^+} \ln|x-2|\Big|_a^3 = \lim_{a\to 2^+} \ln|3-2| - \ln|a-2| = \infty$$
since $\ln|a-2|$ is approaching "$\ln|0| = -\infty$" as $a \to 2$. Since the $\int_2^3$-part diverges, the whole integral diverges. [If we had checked out the $\int_0^2$-part, we would have found it diverges as well. So, unlike the last problem, in this problem *both* pieces diverge.]
(Diverges)

7. $\displaystyle\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln(x)}{x^2}\,dx = \lim_{b\to\infty} -\frac{\ln|x|}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx$
using integration by parts with $u = \ln|x|$ and $dv = x^{-2}dx$.
$$= \lim_{b\to\infty} -\frac{\ln|b|}{b} + 0 + \left(-x^{-1}\right)\Big|_1^b = \lim_{b\to\infty} -\frac{\ln|b|}{b} - \frac{1}{b} + 1 = 0 + 0 + 1$$
Use L'Hopital's rule to evaluate the limit of $\ln|b|/b$.
(Converges to 1)

8. $\displaystyle\int_0^4 \frac{dx}{(x-2)^{5/3}}$
Note: this integral is improper at $x=2$ so we need to split it into 2 pieces. Let's deal with the $\int_0^2$-part first.
$$\int_0^2 \frac{dx}{(x-2)^{5/3}} = \lim_{b\to 2^-} \int_0^b (x-2)^{-5/3}\,dx = \lim_{b\to 2^-} \frac{(x-2)^{-2/3}}{-2/3}\Big|_0^b = \lim_{b\to 2^-} \frac{-3}{2(x-2)^{2/3}}\Big|_0^b$$
$$= \lim_{b\to 2^-} \frac{-3}{2(b-2)^{2/3}} - \frac{-3}{2(0-2)^{2/3}} = \infty$$
Since $(b-2)^{2/3}$ approaches 0 from the left, we get that $1/(b-2)^{2/3}$ approaches $-\infty$. Therefore, because the $\int_0^2$-part diverges, the whole integral diverges. [Again, note that the $\int_2^4$-part diverges as well — we could have used that part to show divergence.]
(Diverges)

Use a **comparison test** to determine if the following integrals converge or diverge.

9. $\displaystyle\int_1^\infty \frac{\cos^2(x)}{1+x^2}\,dx$

   This looks like a p-integral with $p=2$ so we should suspect that it conveges. Thus we need to find an upper bound when comparing. To make a fraction bigger we need to increase the numerator and decrease the denominator.

   $0 \leq \dfrac{\cos^2(x)}{1+x^2} \leq \dfrac{1}{1+x^2} \leq \dfrac{1}{x^2}$. Now note that $\int_1^\infty 1/x^2\,dx$ converges. Thus by the comparison test (for integrals) we conclude that our integral **converges**.

10. $\displaystyle\int_1^\infty \frac{x^3+2x+\sin(x)+1}{3x^4-x-\cos^2(x)}\,dx$

    Ignoring all of the "lower order terms," this integral looks like $\displaystyle\int_1^\infty \frac{x^3}{3x^4}\,dx$ which is essentially an integral like $\int_1^\infty 1/x\,dx$ which diverges (p-integral with $p=1$). So since we suspect that the integral diverges, we should look for a lower bound.

    $\dfrac{x^3+2x+\sin(x)+1}{3x^4-x-\cos^2(x)} \geq \dfrac{x^3}{3x^4-x-\cos^2(x)} \geq \dfrac{x^3}{3x^4} = \dfrac{1}{3x} \geq 0$ (first we decreased the numerator and then increased the denominator). Now we know that $\int_1^\infty 1/x\,dx$ diverges. Thus $\int_1^\infty 1/(3x)\,dx$ diverges too. Therefore, by the comparison test, our integral **diverges**.

11. $\displaystyle\int_1^\infty e^{-x^2}\,dx$ *Hint:* $e^{x^2} \geq e^x$

    If $\int_1^\infty e^{-x}\,dx$ converges, this integral should converge too. $0 < e^{-x^2} = \dfrac{1}{e^{x^2}} \leq \dfrac{1}{e^x} = e^{-x}$ (since we decreased the donominator). [Note: e^x is an increasing function and $x^2 > x$ for $x > 1$. Therefore, $e^{x^2} > e^x$.] Briefly $\displaystyle\int_1^\infty e^{-x}\,dx = -e^{-\infty} + e^{-1} = 1/e$ (converges). Therefore, by the comparison test, our integral **converges**.

12. $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$

    When $0 \leq x \leq 1$ we have that $1 = e^0 \leq e^x \leq e^1 = e$. Therefore, $0 \leq \dfrac{e^{-x}}{\sqrt{x}} = \dfrac{1}{e^x\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$ (we decreased the denominator using $e^x \geq 1$). Next, briefly $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}\,dx$ $2\sqrt{x}\Big|_0^2 = 2\sqrt{1} - 2\sqrt{0}$ (converges). Therefore, by the comparison test, our integral **converges**.