Separation of Variables

Example: Find the general solution of $y' = x^3y^2$.

First, rewrite $y' = \frac{dy}{dx}$ and separate variables: $\frac{dy}{dx} = x^3y^2$ becomes $\frac{dy}{y^2} = x^3\,dx$ (divide by y^2 and multiply by dx).

Next, after noting that $\frac{1}{y^2} = y^{-2}$, integrate both sides: $\displaystyle\int y^{-2}\,dy = \int x^3\,dx$ becomes $\frac{y^{-1}}{-1} + C_1 = \frac{x^4}{4} + C_2$ so that $-\frac{1}{y} = \frac{1}{4}x^4 + C_2 - C_1$. But since C_1 and C_2 are arbitrary constants we can relabel $C_2 - C_1 = C_3$. At this point, $-\frac{1}{y} = \frac{1}{4}x^4 + C_3$ is an implicit general solution.

We seek an *explicit* solution. To this end, negate both side and once again rename the constant: $\frac{1}{y} = -\frac{1}{4}x^4 - C_3 = -\frac{1}{4}x^4 + C_4$. Now take reciprocals and get $y = \dfrac{1}{C_4 - \frac{x^4}{4}}$. We can simplify the fraction a bit more by multiplying by $1 = \frac{4}{4}$ to get: $y = \dfrac{4}{4C_4 - x^4}$. Finally, rename our constant ($4C_4 = C$) one last time to get: $\boxed{y = \dfrac{4}{C - x^4}}$.

Example: Solve the initial value problem (IVP): $y' = \frac{2x+1}{y}$ where $y(0) = -5$.

First, rewrite $y' = \frac{dy}{dx}$ and separate variables: $\frac{dy}{dx} = \frac{2x+1}{y}$ becomes $y\,dy = (2x+1)\,dx$. Then integrate: $\frac{y^2}{2} = \int y\,dy = \int (2x+1)\,dx = x^2 + x + C_1$. Now solve for y (renaming the constant along the way): $y^2 = 2x^2 + 2x + C$ and so $y = \pm\sqrt{2x^2 + 2x + C}$ (this is an explicit general solution of our differential equation).

Now we consider the initial value: $-5 = y(0) = \pm\sqrt{2(0^2) + 2(0) + C} = \pm\sqrt{C}$. We see that we must choose the minus sign in $\pm$. We have $-5 = -\sqrt{C}$ so that $25 = C$. Therefore, the solution of our IVP is $\boxed{y = -\sqrt{2x^2 + 2x + 25}}$.

Example: A certain population grows at a rate proportional to its size. It doubles in size every 5 years and there are initially 123 people. Find a formula for $P(t)$, the population after t years have gone by.

If $P(t)$ is our population t years from now, $P'(t)$ is its growth rate (in people per year). Therefore, if $P(t)$ grows at a rate proportional to its size, we have $P'(t) \propto P(t)$. In other words, there is some constant k (called a *growth constant*) such that $P' = kP$. We can solve this separably variable equation: $\frac{dP}{dt} = kP$ becomes $\frac{dP}{P} = k\,dt$ so $\ln|P| = \int \frac{1}{P}\,dP = \int k\,dt = kt + C_1$. Exponentiate and get $|P| = e^{\ln|P|} = e^{kt+C_1} = e^{kt}e^{C_1}$. Thus $P(t) = \pm e^{C_1}e^{kt}$. Now we acknowledge that since e^{C_1} can be any positive real number, $\pm e^{C_1}$ can be any nonzero real number. We also, note that $P(t) = 0$ is a solution of our differential equation (we lost this solution when separating variables and dividing by P). Long story short we can rename $\pm e^{C_1}$ (along with 0) with C. Therefore, $P(t) = Ce^{kt}$ is a general solution of our differential equation (DE).

Next, let's see what the doubling population tells us. Notice that $P(0) = Ce^{k(0)} = Ce^0 = C$ and we should have twice that population in 5 years, so $2C = P(5) = Ce^{k5}$. Therefore, $2 = e^{5k}$ and so $\ln(2) = 5k$. Thus our growth constant is $k = \frac{1}{5}\ln(2)$. Plugging this into our general solution, we have $P(t) = Ce^{\frac{1}{5}\ln(2)\cdot t} = Ce^{\ln(2)\cdot\frac{t}{5}} = C(e^{\ln(2)})^{t/5}$. Therefore, $P(t) = C2^{t/5}$.

We also know that the initial population is $P(0) = 123$ people. Thus $123 = P(0) = C2^{0/5} = C2^0 = C$. Therefore, the solution of our IVP is $\boxed{P(t) = 123 \cdot 2^{t/5}}$.

Example: Newton's Law of Cooling: Suppose we pour a cup of coffee and it is 150°F initially and then 100°F after 10 minutes in a 75°F room. Find a formula for the temperature of the coffee, $T(t)$, in degrees Fahrenheit t minutes after it was poured.

Newton's law of cooling says that an object will heat up or cool at a rate proportional to the difference between its temperature and the ambient temperature. In other words: $T'(t) \propto T(t) - R$ if R is our ambient room temperature. Thus there is a heating/cooling constant k such that $T' = k(T - R)$. In our situation, $R = 75$.

We can separate $\frac{dT}{dt} = k(T - 75)$ and solve as follows: $\frac{dT}{T-75} = k\,dt$ so that $\ln|T - 75| = \int \frac{dT}{T-75} = \int k\,dt = kt + C_1$.[1] Exponentiate, drop absolute values, acknowledge a lost solution ($T(t) = 75$), and rename our constant: $T(t) = 75 + Ce^{kt}$.

Next, we have initially $150 = T(0) = 75 + Ce^{k\cdot 0} = 75 + C$ so that $C = 75$. Also, after 10 minutes $100 = T(10) = 75 + 75e^{k10}$. Thus $25 = 75e^{10k}$ so $\frac{1}{3} = e^{10k}$ and so $\ln(1/3) = 10k$. Therefore, $k = \frac{1}{10}\ln(1/3) = \frac{1}{10}\ln(3^{-1}) = -\frac{1}{10}\ln(3)$. We now have $T(t) = 75 + 75e^{-\frac{1}{10}\ln(3)\cdot t} = 75 + 75(e^{\ln(3)})^{-t/10} = 75 + 75\cdot 3^{-t/10}$. Therefore, the solution of our IVP is $\boxed{T(t) = 75(1 + 3^{-t/10})}$.

[1] The integration on the left is done with a simple substitution. Let $u = T - 75$ so that $du = dT$. Thus $\int \frac{dT}{T-75} = \int \frac{du}{u} = \ln|u| + C = \ln|T - 75| + C$.