Logistic Growth

When populations are small relative to their environment and resources, population growth tends to be proportional to the size of the population. In other words, if $P(t)$ is a population and t is time, we expect to see $P'(t) \approx k \cdot P(t)$ for some growth constant k when $P(t)$ is relatively small.

However, when the population starts getting big relative to its environment and resources become scarce (e.g., food or land), we expect to see this growth to come to a near stop. Imagine we are situated in a region that can only reasonably sustain a population of M (i.e., the location's *carrying capacity*). Then as $P(t)$ approaches M, we should see $P'(t) \approx 0$.

A simple model that incorporates both of these features is *logistic growth*. Here we have $P'(t) = kP\left(1 - \frac{P}{M}\right)$ and let $P(0) = P_0$ denote some initial population. Notice that if P is much smaller than M, $1 - \frac{P}{M} \approx 1 - 0 = 1$ so $P' \approx kP$ and if P is close to M, then $1 - \frac{P}{M} \approx 1 - 1 = 0$ so $P' \approx 0$.

We now solve the logistic equation using separation of variables and a partial fraction decomposition. We have $\frac{dP}{dt} = kP(1 - P/M)$ so that $\frac{dP}{P(1-P/M)} = k\,dt$. We then seek to integrate both sides. The right hand side is easy: $\int k\,dt = kt + C_1$ for some constant C_1. The left hand side requires a partial fraction decomposition.

We have $\dfrac{1}{P(1-P/M)} = \dfrac{M}{M} \cdot \dfrac{1}{P(1-P/M)} = \dfrac{M}{P(M-P)} = \dfrac{-M}{P}P - M = \dfrac{A}{P} + \dfrac{B}{P-M}$ for some A and B. Clearing the denominators (i.e., multiplying both sides by $P(P-M)$), we get $-M = A(P-M) + B \cdot P$. Now plug in our roots. At $P = 0$, we get $-M = A(-M) + 0$ so $A = 1$ and at $P = M$, we get $-M = 0 + B \cdot M$ so $B = -1$.

Therefore, we integrate $\displaystyle\int \left(\frac{1}{P} + \frac{-1}{P-M}\right) dP = \ln|P| - \ln|P-M|$ (plus some constant). Putting this together with our other (easy) integral, we get $\ln|P| - \ln|P-M| = kt + C_1$. Next, use a law of logs: $\ln\left|\dfrac{P}{P-M}\right| = kt + C_1$. Then exponentiate: $\left|\dfrac{P}{P-M}\right| = e^{kt+C_1} = e^{kt}e^{C_1}$ and drop absolute values: $\dfrac{P}{P-M} = \pm e^{C_1}e^{kt}$. Acknowledge that we lost a zero solution when dividing during the separation of variables and so $\pm e^{C_1}$ (plus that zero solution) account for all possible constants, so we rename this as C. Therefore, $\dfrac{P}{P-M} = Ce^{kt}$.

Next, we solve for P. We have $P = Ce^{kt}(P-M) = PCe^{kt} - MCe^{kt}$ so $-MCe^{kt} = P - PCe^{kt} = P(1 - Ce^{kt})$. Thus $P = \dfrac{-MCe^{kt}}{1 - Ce^{kt}} = \dfrac{MCe^{kt}}{Ce^{kt} - 1} = \dfrac{Ce^{kt}}{Ce^{kt}} \cdot \dfrac{M}{1 - Ce^{-kt}}$. Therefore, our general solution is $P(t) = \dfrac{M}{1 - Ce^{-kt}}$ (as well as $P(t) = 0$ which was another solution that we lost due to an earlier division).

Finally, assuming that $P(0) = P_0$, we have $P_0 = \dfrac{M}{1 - Ce^0} = \dfrac{M}{1-C}$ so that $M = P_0(1-C) = P_0 - P_0 \cdot C$. Thus $P_0 \cdot C = P_0 - M$ and so $C = \dfrac{P_0 - M}{P_0}$. Therefore, $P(t) = \dfrac{M}{1 - (P_0 - M)/P_0 e^{-kt}} = \dfrac{M}{1 - (P_0 - M)/P_0 e^{-kt}} \cdot \dfrac{P_0}{P_0}$. We then arrive at our solution's final form: $\boxed{P(t) = \dfrac{M \cdot P_0}{P_0 - (P_0 - M)e^{-kt}}}$.

Note that if $P_0 = 0$, then $P(t) = 0$ (no population stays that way). Also, if $P_0 = M$, then $P(t) = M$ (if we are already at our carrying capacity, we stay there). Also, notice that, assuming $k > 0$ (i.e., we have a *growth* constant), then $e^{-kt} \to 0$ as $t \to \infty$. Thus $P(t) \to \dfrac{M \cdot P_0}{P_0 - 0} = M$ as $t \to \infty$. In other words, we should expect our population to tend toward our carrying capacity regardless of where it started. So a small population over a long time will grow to approximately M and a very large population over a long time will shrink to M.