Math 1120

Series Tests

Mixed Practice

#1. $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$

#2. $\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$

#3. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

#4. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n-1}{n^2+n}$

#5. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

#6. $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

#7. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$

#8. $\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

#9. $\sum_{k=1}^{\infty} k^2 e^{-k}$

#10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

#11. $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln(n)}$

#12. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+25}$

#13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

#14. $\sum_{n=1}^{\infty} \sin(n)$

#15. $\sum_{n=0}^{\infty} \frac{n!}{2\cdot 5\cdot 8\cdot \cdots \cdot (3n+2)}$

#16. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

#17. $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$

#18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

#19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$

#20. $\sum_{k=1}^{\infty} \frac{k+5}{5^k}$

#21. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

#22. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

#23. $\sum_{n=1}^{\infty} \tan(1/n)$

#24. $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$

#25. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

#26. $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$

#27. $\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3}$

#28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

#29. $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n\sqrt{n}}$

#30. $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$

#31. $\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$

#32. $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

#33. $\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$

#34. $\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2(n)}$

#35. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

#36. $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$

#37. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)^n$

#38. $\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$

Answers with suggested tests: There are a few places where you might be able to avoid the integral test by using a ratio test or a clever comparison. In other words, I do not claim these solutions are optimal. For conditionally convergent series the suggested tests are given for the *original series* (converges) plus *related absolute values series* (diverges).

Abbreviations: Div. = Diverges, C. Conv. = Conditionally Convergent, Conv. Abs. = Converges Absolutely, geom. = geometric series, $p = X = p$-series with exponent X, harm. = harmonic series, comp. = comparison (usually limit comparison is intended), dir. comp. = direct comparison, AST = alternating series test, and int. test = integral test.

#1. Div. (n^{th} term)

#2. Div. (comp. harm.)

#3. Conv. Abs. (comp. $p = 2$)

#4. C. Conv. (AST plus comp. harm.)

#5. Conv. Abs. (geom. r=-3/8)

#6. Conv. Abs. (root)

#7. Div. (int. test)

#8. Conv. Abs. (ratio)

#9. Conv. Abs. (ratio)

#10. Conv. Abs. (ratio or int test)

#11. C. Conv. (AST plus int. test)

#12. C. Conv. (AST plus comp. harm.)

#13. Conv. Abs. (ratio)

#14. Div. (n^{th} term)

#15. Conv. Abs. (ratio)

#16. Div. (comp. harm.)

#17. Div. (n^{th} term)

#18. C. Conv. (AST plus comp. $p = 0.5$)

#19. C. Conv. (AST plus dir. comp. $p = 0.5$)

#20. Conv. Abs. (ratio)

#21. Conv. Abs. (root)

#22. Conv. Abs. (comp. $p = 2$)

#23. Div. (comp. harm.) [solution below]

#24. Conv. Abs. (dir. comp. $p = 2$)

#25. Conv. Abs. (ratio)

#26. Conv. Abs. (ratio)

#27. Conv. Abs. (comp. $p = 1.5$)

#28. Conv. Abs. (dir. comp. $p = 2$)

#29. Conv. Abs. (dir. comp. $p = 1.5$) Note: $\arctan(x) \leq \pi/2$

#30. C. Conv. (AST plus comp. $p = 0.5$)

#31. Div. (ratio or n^{th} term)

#32. Conv. Abs. (root)

#33. Conv. Abs. (dir. comp. $p = 1.5$) [solution below]

#34. Div. (dir. comp. harm.)

#35. Div. (n^{th} term)

#36. Conv. Abs. (tricky dir. comp. $p = 2$)

#37. Conv. Abs. (root)

#38. Div. (dir. comp. harm.) [solution below]

#23: The linearization (i.e., tangent) of $\tan(x)$ centered at $x = 0$ is just x. So for small x's (i.e., $1/n$ for large n), we have ($\tan(x) \approx x$) (i.e., $\tan(1/n) \approx 1/n$). This rather non-obvious observation leads us to compare with the harmonic series. We have $L = \lim_{n\to\infty} \dfrac{\tan(1/n)}{1/n} = \lim_{t\to 0^+} \dfrac{\tan(t)}{t} = \lim_{t\to 0^+} \dfrac{\sec^2(t)}{1} = \sec^2(0) = 1$ (using the fact that $n \to \infty$ means $t = 1/n \to 0^+$ as well as an application of L'Hopital's rule). Therefore, we get $0 < L = 1 < \infty$, so the limit comparison test says that series #23 diverges because the harmonic series $\sum 1/n$ diverges.

#36: A non-obvious observation: $(\ln(n))^{\ln(n)} = e^{\ln\left((\ln(n))^{\ln(n)}\right)} = e^{\ln(n)\cdot\ln(\ln(n))} = e^{\ln(\ln(n))\cdot\ln(n)} = e^{\ln\left(n^{\ln(\ln(n))}\right)} = n^{\ln(\ln(n))}$. Next, $\ln(\ln(n)) \geq 2$ for large enough n (Why? $\ln(\ln(\infty)) = \ln(\infty) = \infty$). Therefore, for large enough n, $n^2 \leq n^{\ln(\ln(n))}$. Thus for such n, $\frac{1}{n^{\ln(\ln(n))}} \leq \frac{1}{n^2}$. Therefore, by direct comparison with the convergent p-series ($p = 2 > 1$), series #36 must converge.

#38: This one is tricky. Since $\sqrt[n]{2} - 1 = 2^{1/n} - 1 \to 2^{1/\infty} - 1 = 2^0 - 1 = 0$, the n^{th} term test does not help. One can show that $2^{1/n} = e^{\ln(2^{1/n})} = e^{\frac{\ln(2)}{n}} > 1 + \frac{\ln(2)}{n}$. [This is most easily seen from the exponential function's Maclaurin series – which we don't know about yet.] Anyway, this means that $2^{1/n} - 1 > \frac{\ln(2)}{n}$ so series #38 diverges by direct comparison with ($\ln(2)$ times) the harmonic series.