A quick introduction to Lie and vertex algebras

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Outline:

The Classical Associative and Lie Algebras

The Non-Classical Vertex Algebras

A Non-Trivial Example the $\widehat{\mathfrak{sl}}_2$-module $L(\Lambda_0)$

Identities A character formula and combinatorial identities

Classical Algebras:

Associative Algebras
and
Lie Algebras

Algebras (over $\mathbb{F}$):

An Algebra, $\mathcal{A}$, is a vector space (over some field $\mathbb{F}$) equipped with a multiplication map: $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that m is bilinear.

Multiplication is *usually* denoted by juxtaposition: $m(u, v) = uv$.

Examples:

Real Matrix Algebras Let $\mathcal{A} = \mathbb{R}^{n \times n}$ be all $n \times n$ matrices with real entries.

Polynomial Algebras Let $\mathcal{A} = \mathbb{R}[x, y]$ be all polynomials in two indeterminants (x and y) with real coefficients.

Cross Product Algebra Let $\mathcal{A} = \mathbb{R}^3$ here we multiply vectors by taking their cross product: $u \times v$.

Special Properties:

Let $\mathcal{A}$ be an algebra (over some field $\mathbb{F}$).

Associative $\mathcal{A}$ is Associative if $u(vw) = (uv)w$ for all $u, v, w \in \mathcal{A}$.

Unital $\mathcal{A}$ is Unital or an algebra with identity if there exists some $\mathbf{1} \in \mathcal{A}$ such that $\mathbf{1}v = v\mathbf{1} = v$ for all $v \in \mathcal{A}$.

Commutative $\mathcal{A}$ is Commutative if $uv = vu$ for all $u, v \in \mathcal{A}$.

Examples:

- $\mathcal{A} = \mathbb{R}[x, y]$ is a commutative associative unital algebra. It's identity is the polynomial 1.
- $\mathcal{A} = \mathbb{R}^{n \times n}$ is an associative unital algebra, but it is *not* commutative (unless $n = 1$). It's identity is the *identity matrix* I_n.
- $\mathcal{A} = \mathbb{R}^3$ equipped with the cross product is a non-associative non-commutative algebra and has no identity. So what kind of algebra is this?

Lie Algebras:

Let L be an algebra (over some field $\mathbb{F}$). Instead of using juxtaposition, let's denote multiplication with a *bracket*: $m(u, v) = [u, v]$. L is called a Lie Algebra if the following axioms hold:

Skew-Commutative $[v, v] = 0$ for all $v \in L$.

Jacobi Identity $[[u, v], w] + [[v, w], u] + [[w, u], v] = 0$ for all $u, v, w \in L$.

Examples:

- $\mathcal{A} = \mathbb{R}^3$ equipped with the cross product is a Lie algebra. Remember that $v \times v = 0$ (the cross product of parallel vectors is zero). A tedious calculation shows that the Jacobi identity holds as well.

- If we give the matrix algebra $\mathbb{R}^{n \times n}$ a different multiplication, called the commutator bracket, defined by $[A, B] = AB - BA$, it becomes a Lie algebra. To remind ourselves that we are using a different "multiplication" we call this algebra $\mathfrak{gl}(n, \mathbb{R})$.

Lie Algebras:

What do the axioms *really* say?

- The first axiom, $[v, v] = 0$, implies the following:

  $0 = [u+v, u+v]$

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  Therefore, $[u, v] = -[v, u]$ (almost commutative!)

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  Therefore, $[u,v] = -[v,u]$ (almost commutative!)

- The Jacobi identity says much more. Using the above property we can re-write the Jocobi identity as follows:

$$0 = [[u,v],w] + [[v,w],u] + [[w,u],v]$$

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$$[u, [v, w]] = [[u, v], w] + [[w, u], v]$$

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$$[u, [v, w]] = [[u, v], w] - [v, [w, u]]$$

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$$[u,[v,w]] = [[u,v],w] + [v, -[w,u]]$$

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$$[u, [v, w]] = [[u, v], w] + [v, [u, w]]$$

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  Therefore, $[u,v]=-[v,u]$ (almost commutative!)

- The Jacobi identity says much more. Using the above property we can re-write the Jocobi identity as follows:

$$\begin{array}{ccccc} [u,[v,w]] & = & [[u,v],w] & + & [v,[u,w]] \\ \frac{d}{dt}\left[f(t)g(t)\right] & = & \frac{d}{dt}\left[f(t)\right]g(t) & + & f(t)\frac{d}{dt}\left[g(t)\right] \end{array}$$

  Let $\mathcal{A}$ be an algebra and $\partial:\mathcal{A}\to\mathcal{A}$. If $\partial(uv)=\partial(u)v+u\partial(v)$ for all $u,v\in\mathcal{A}$, then we say that ∂ is a derivation of $\mathcal{A}$.

  The Jacobi identity simply says,
  "The multiplication operators of L are derivations."

Non-Classical Algebras:
Vertex (Operator) Algebras

Origins of Vertex Operator Algebras:

1970s Vertex operators appear in the study of String Theory.

1980s Mathematicians use vertex operators to study certain representations of affine Lie algebras.

1984 I. Frenkel, J. Lepowsky, and A. Meurman construct $V^\natural$.

1986 R. Borcherds introduces a set of axioms for a notion which he calls a "vertex algebra".

1992 R. Borcherds proves the Conway-Norton conjectures.

The Definition of a Vertex Algebra

Algebras (over $\mathbb{C}$):

Let V be a vector space over $\mathbb{C}$.

Vertex Algebras:

Let V be a vector space over $\mathbb{C}$.

Algebras (over $\mathbb{C}$):

Equip V with a bilinear map $\cdot : V \times V \to V$ (a multiplication map).

$(u, v) \mapsto u \cdot v$

Vertex Algebras:

Equip V with infinitely many bilinear maps $n : V \times V \to V$ (where $n \in \mathbb{Z}$).

$(u, v) \mapsto u_n v$

Algebras (over $\mathbb{C}$):

Equip V with a bilinear map $\cdot : V \times V \to V$ (a multiplication map).

$(u, v) \mapsto u \cdot v$

Vertex Algebras:

Equip V with a bilinear map $Y(\cdot, x) : V \times V \to V[[x, x^{-1}]]$ ($V[[x, x^{-1}]]$ are Laurent series with coefficients in V).

$$(u, v) \mapsto Y(u, x)v = \sum_{n \in \mathbb{Z}} u_n v \, x^{-n-1}$$

Notation:

$V[x]$ polynomials in x with coefficients in V.
$V[x, x^{-1}]$ Laurent polynomials in x with coefficients in V.
$V[[x]]$ power series in x with coefficients in V.
$V((x))$ lower truncated Laurent series with coefficients in V.
$V[[x, x^{-1}]]$ Laurent series in x with coefficients in V.

WARNING: $\mathbb{C}[[x, x^{-1}]]$ **is not an algebra!** Sometimes mutiplication isn't well defined.

Example: $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$ is the formal delta function. Notice that $(\delta(x))^2$ is undefined.

Algebras (over $\mathbb{C}$):

Equip V with a bilinear map $\cdot : V \times V \to V$ (a multiplication map).

$(u, v) \mapsto u \cdot v$

Vertex Algebras:

Equip V with a bilinear map $Y(\cdot, x) : V \times V \to V((x))$
($V((x))$ are lower truncated Laurent series with coefficients in V).

$$(u, v) \mapsto Y(u, x)v = \sum_{n \in \mathbb{Z}} u_n v \, x^{-n-1}$$

where $u_n v = 0$ for $n \gg 0$.

Algebras (over $\mathbb{C}$):

To be a Lie algebra V's multiplication must be skew-symmetric and satisfy the Jacobi identity.

$(uv)w + (vw)u + (wu)v = 0$

Vertex Algebras:

V's vertex operators must satisfy the Jacobi identity.

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2)w - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)Y(v, x_2)Y(u, x_1)w$$

$$= x_2^{-1}\delta\left(\frac{x_1 - x_0}{-x_2}\right)Y(Y(u, x_0)v, x_2)w$$

Algebras (over $\mathbb{C}$):

To be a Lie algebra V's multiplication must be skew-symmetric and satisfy the Jacobi identity.

$\cancel{(uv)w + (vw)u + (wu)v = 0}$
$u(vw) = (uv)w + v(uw)$ (the product rule)

Vertex Algebras:

V's vertex operators must satisfy the Jacobi identity.

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)(Y(v, x_2)w) =$$

$$x_2^{-1}\delta\left(\frac{x_1 - x_0}{-x_2}\right) Y(Y(u, x_0)v, x_2)w + x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)(Y(u, x_1)w)$$

Algebras (over $\mathbb{C}$):

To be a unital algebra V must have an identity vector $\mathbf{1}$.

$$\mathbf{1}u = u\mathbf{1} = u$$

Vertex Algebras:

A vertex algebra has a vaccuum vector $\mathbf{1}$.

$$Y(\mathbf{1}, x)u = u \qquad \text{and} \qquad Y(u, x)\mathbf{1} = e^{x\mathcal{D}}u$$

In particular, $Y(u, 0)\mathbf{1} = u$. [Note: $\mathcal{D}(u) = u_{-2}\mathbf{1}$]

Algebras (over $\mathbb{C}$):

V is a commutative algebra if...

$$uv = vu$$

Vertex Algebras:

A vertex operators satisfy a property called locality.

$$(x_1 - x_2)^N Y(u, x_1) Y(v, x_2) = (x_1 - x_2)^N Y(v, x_2) Y(u, x_1)$$

for some $N \gg 0$

Algebras (over $\mathbb{C}$):

V is an associative algebra if...

$$(uv)w = u(vw)$$

Vertex Algebras:

A vertex operators satisfy a property called weak associativity.

$$(x_1 - x_2)^N Y((Y(u, x_1)v), x_2)w = (x_1 + x_2)^N Y(u, x_1 + x_2)(Y(v, x_2)w)$$

for some $N \gg 0$

Summary: vertex algebras generalize both Lie algebras and commutative associative unital algebras.

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...which is interesting since $0 = [1, 1] = 1$ so that...

$$u = [u, 1] = [u, 0] = 0$$

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...which is interesting since $0 = [1, 1] = 1$ so that...

$$u = [u, 1] = [u, 0] = 0$$

Classically speaking, these structures don't go together at all!

A “Trivial” Example:

Holomorphic Vertex Algebras

Let $\mathcal{A} = \mathbb{C}[t]$ be the commutative associative unital algebra of complex polynomials and let $D : \mathcal{A} \to \mathcal{A}$ be the derivation $D = \dfrac{d}{dt}$.

Define $Y(u, x) = e^{xD}u$. That is:

$$Y(f(t), x) = e^{x\frac{d}{dt}} f(t) = f(t + x)$$

So...

$$f(t)_{-n-1}g(t) = \text{ coefficient of } x^n \text{ in...}$$

$$Y(f(t), x)g(t) = f(t + x)g(t).$$

$\mathcal{A} = \mathbb{C}[t]$ equipped with this vertex operator map, $Y(\cdot, x)$, is a vertex algebra with vacuum vector 1.

A “Trivial” Example:

Holomorphic Vertex Algebras

Let $\mathcal{A}$ be a commutative associative unital algebra (whose identity is $\mathbf{1}$) and let $D : \mathcal{A} \to \mathcal{A}$ be a derivation of $\mathcal{A}$.

Define $Y(u, x) = e^{xD}u$. That is:

$$Y(u,x)v = \sum_{n=0}^{\infty} u_{-n-1}vx^n \qquad \text{where} \qquad u_{-n-1}v = \frac{1}{n!}D^n(u)v$$

Then, $\mathcal{A}$ equipped with this vertex operator map, $Y(\cdot, x)$, becomes a vertex algebra with vacuum vector $\mathbf{1}$.

Note: If $D = 0$, we simply have $Y(u,x)v = uv$. That is:
All commutative associative unital algebras are vertex algebras.

An Example: $\widehat{\mathfrak{sl}}_2$ and the $\widehat{\mathfrak{sl}}_2$-module $L(\Lambda_0)$

The 3-dimensional simple Lie algebra $\mathfrak{sl}_2$:

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ A \in \mathbb{C}^{2\times 2} \,|\, \mathrm{tr}(A) = 0 \right\}$$

The vector space $\mathfrak{sl}_2$ becomes a Lie algebra when given the commutator bracket $[A, B] = AB - BA$.

Consider the following basis for $\mathfrak{sl}_2$:

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The commutator brackets among the basis elements are:

$$[E, F] = H \qquad [H, E] = 2E \qquad [H, F] = -2F$$

(use skew-symmetry to find the rest).

Affine $\mathfrak{sl}_2$:

The infinite dimensional Lie algebra $\widehat{\mathfrak{sl}}_2$ is a central extension of the loop algebra over $\mathfrak{sl}_2$.

As a vector space, $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2[t, t^{-1}] \oplus \mathbb{C}c$.

The element c is central: $[c, x] = 0$ for all $x \in \widehat{\mathfrak{sl}}_2$.

Let $a, b \in \mathfrak{sl}_2$ and $m, n \in \mathbb{Z}$

$$[a\, t^m, b\, t^n] = [a, b]\, t^{m+n} + m\, \mathrm{tr}(ab)\delta_{m+n,0}c$$

Let $\mathfrak{b} = \mathfrak{sl}_2[t] \oplus \mathbb{C}c$. Notice that $\mathfrak{b}$ is a subalgebra of $\widehat{\mathfrak{sl}}_2$.

The $\widehat{\mathfrak{sl}}_2$-module $L(\Lambda_0)$.

Let $\mathbb{C}_1 = \mathbb{C}\mathbf{1}$ be a 1-dimensional $\mathfrak{b}$-module with module action defined as follows:

- $c \cdot \mathbf{1} = \mathbf{1}$
- $a\, t^n \cdot \mathbf{1} = 0$ for all $a \in \mathfrak{sl}_2$ and $n \in \mathbb{Z}_{\geq 0}$

We define the $\widehat{\mathfrak{sl}}_2$-module V to be

$$V = U(\widehat{\mathfrak{sl}}_2) \otimes_{U(\mathfrak{b})} \mathbb{C}_1.$$

V is an example of a Verma module.

Finally, let $L(\Lambda_0) = V/J$ where J is the maximal proper submodule of V (this implies that $L(\Lambda_0)$ is an irreducible module).

Note: We will denote the action of $a\, t^n$ on $L(\Lambda_0)$ by $a(n)$.

Collect all operators $a(n)$ (where $a \in \mathfrak{sl}_2$ and $n \in \mathbb{Z}$) together in a generating function:

$$a(x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1}$$

For each $n \in \mathbb{Z}$ define the following n^{th}-product:

$$a(x)_n b(x) = \mathrm{Res}_{x_1} \left((x_1 - x)^n a(x_1) b(x) - (-x + x_1)^n b(x) a(x_1) \right)$$

Finally define a linear map $Y(\cdot, x)$ as follows:

$$Y(a^{(1)}(n_1) a^{(2)}(n_2) \ldots a^{(r)}(n_r) \mathbf{1}, x) = a^{(1)}(x)_{n_1} a^{(2)}(x)_{n_2} \ldots a^{(r)}(x)_{n_r} \mathbf{1}$$

for $r \geq 0$, $a^{(i)} \in \hat{\mathfrak{sl}}_2$ and $n_i \in \mathbb{Z}$.

Notice that for $a \in \mathfrak{sl}_2$ we have:

$$Y(a, x) = a(x) = \sum_{n \in \mathbb{Z}} a(n) x^{-n-1}$$

Thm: V is a vertex (operator) algebra, J is an ideal of V, and $L(\Lambda_0) = V/J$ is a simple vertex (operator) algebra.

Combinatorial Identities

Define $\Delta(H,x) = x^{H(0)} \exp\left(\sum_{k=1}^{\infty} \frac{H(k)}{-k}(-x)^{-k}\right)$.

Thm [Li]: $(L(\Lambda_0), Y(\cdot,x))$ is isomorphic to $(L(\Lambda_0), Y(\Delta(H,x)\cdot,x))$ as an $L(\Lambda_0)$-module.

Notation: Let $v \in L(\Lambda_0)$.

$$Y(\Delta(H,x)v,x) = \sum_{m\in\mathbb{Z}} v_{(H)}(m)x^{-m-1}$$

and

$$Y(\Delta(H,x)\omega,x) = \sum_{m\in\mathbb{Z}} L_{(H)}(m)x^{-m-2}$$

Def: The character of $L(\Lambda_0)$ is given by:

$$\chi(x;q) = \mathrm{tr}_{L(\Lambda_0)} x^{\frac{1}{2}H(0)} q^{L(0)}$$

Lemma:

$$\begin{aligned}
\left(\frac{1}{2}H\right)_{(H)}(0) &= \frac{1}{2}H(0) + 1 \\
L_{(H)}(0) &= L(0) + H(0) + 1 \\
&= L(0) + 2\left(\frac{1}{2}H(0)\right) + 1
\end{aligned}$$

Recurrence Thm:

$$\chi(x;q) = (xq)\chi(xq^2;q)$$

Solving the recurrence relations obtained from our Recurrence Theorem and plugging in an initial condition found by analyzing a “lattice” vertex algebra construction of $L(\Lambda_0)$, we obtain the following formula for the character of $L(\Lambda_0)$:

Cor:

$$\chi(x;q) = \prod_{j=1}^{\infty} (1-q^j)^{-1} \sum_{n\in\mathbb{Z}} q^{n^2} x^n$$

The principal character is easily obtained from the "full" character.

Prop: $\chi^P(q) = \chi(q^{-1}; q^2)$.

Cor: We have the following multi-sum principal character formula...

$$\chi^P(q) = \prod_{j\geq 1}(1-q^{2j})^{-1}\sum_{n\in\mathbb{Z}} q^{2n^2-n}$$

Using the theory of affine Lie algebras Kac obtains the following product principal character formula...

$$\chi^P(q) = \prod_{j \geq 1} \frac{(1-q^{2j})}{(1-q^j)}$$

Combining this with our principal character formula, we obtain:

$$\prod_{j \geq 1} \frac{(1-q^{2j})}{(1-q^j)} = \prod_{j \geq 1} (1-q^{2j})^{-1} \sum_{n \in \mathbb{Z}} q^{2n^2-n}$$

Rearranging terms a little bit gives us the following identity (credited to Gauss):

$$\prod_{j \geq 1} \frac{(1-q^{2j})^2}{(1-q^j)} = \sum_{n \in \mathbb{Z}} q^{2n^2-n}$$