Math 2110 **Homework #10** **Due: Fri., Nov. 13^{th}, 2020**

Handwritten solutions are fine. ☺

Notational Notes: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are natural numbers, integers, rational numbers, and real numbers (respectively).

#1 Direct cardinality.

(a) Show that $|\mathbb{N}^2| \leq |S|$ where $S = \{A \in \mathcal{P}(\mathbb{N}) \mid A \text{ is finite}\}$ by finding an injection (i.e., one-to-one function) from $\mathbb{N}^2$ to S. *Note:* You should prove your function is one-to-one.

Warning: One might think of a function like $f(a,b) = \{a,b\}$ (this sends an ordered pair to a finite set). But this function is not one-to-one: $f(1,2) = \{1,2\}$ but $f(2,1) = \{2,1\} = \{1,2\}$ as well. Suggestion: make sure you send a and b to distinct things – think even vs. odd.

(b) Show $\mathbb{O} = \{2k+1 \mid k \in \mathbb{Z}\} = \{\dots,-3,-1,1,3,5,\dots\}$ is countably infinite by finding a bijection (i.e., invertible function) between $\mathbb{O}$ and $\mathbb{N}$. *Note:* You should prove your function is invertible (or one-to-one and onto).

#2 Calculate some cardinalities. Justify your answers using basic cardinal arithmetic and inequalities.

Note: Your answers should either be a non-negative integer (0 or 7 or etc.), $\aleph_0$, $\mathfrak{c} = 2^{\aleph_0}$, or $2^{\mathfrak{c}} = 2^{2^{\aleph_0}}$.

For example, $|\mathbb{Z}^{\mathbb{Z}}| = \mathfrak{c}$ $(= 2^{\aleph_0})$ since $|\mathbb{Z}^{\mathbb{Z}}| = |\mathbb{Z}|^{|\mathbb{Z}|} = \aleph_0^{\aleph_0}$ and $2^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0}$ so that $\aleph_0^{\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$.

(a) $|10^{\mathcal{P}(\mathbb{Z})}|$

(b) $|\mathbb{Z} \times \mathbb{R}^3| + |\mathbb{Q}^{\mathbb{N}}|$

(c) Consider $\mathbb{N}^5$ vs. $5^{\mathbb{N}}$. Do these sets have the same cardinality? Are they countable? (both, either, neither?)

#3 Define the relation $R = \{(a,b) \in \mathbb{Z} \mid \max\{a,b\} = a\}$.

For example: We have $3\,R\,2$ since $\max\{3,2\} = 3$. On the other hand, $7 \not R\, 11$ since $\max\{7,11\} = 11 \neq 7$.

(a) Consider the properties: reflexive, symmetric, anti-symmetric, and transitive. If a property holds, prove it. If it fails, give a counter-example.

(b) Is R a partial order? A total order? An equivalence relation? Explain why or why not for each.

#4 A Modular Arithmetic Problem.

(a) Denote the equivalence class of an integer k with respect to the equivalence relation "mod n" by $[k]$.

First, list all of the equivalence classes of integers with respect to the equivalence relation "mod 2" (show their contents – don't just write "[0]" etc.).

Next, do the same for "mod 6".

Finally, prove that $f : \mathbb{Z}_2 \to \mathbb{Z}_6$ where $f([x]) = [3x]$ is a well-defined function.

(b) Let n be a positive integer. Prove that multiplication mod n is well defined.

In particular, for any $a,b,c,d \in \mathbb{Z}$ where $a \equiv c \pmod{n}$ and $b \equiv d \pmod{n}$ show that $ab \equiv cd \pmod{n}$.