Math 2110 **Homework #5** **Due: Wed., Sept. 30^{th}, 2020**

Handwritten solutions are fine. ☺

#1 In class we showed $2 + 2 = 4$ and $3 \cdot 2 = 6$ more-or-less using Axioms 11–14 on our E and PA handout.

(a) Show $4 + 3 = 7$.

(b) Show $5 \cdot 3 = 15$.

Use the same formatting and level of detail as my "proofs" shown below. You may "bury the details" of addition when working through the multiplication problem.

Showing $2 + 2 = 4$:

1: $2 + (1')$	Definition of $2 = 1'$
2: $(2 + 1)'$	Axiom 12
3: $(2 + 0')'$	Definition of $1 = 0'$
4: $((2 + 0)')'$	Axiom 12
5: $((2)')'$	Axiom 11
6: 4	Definition of $4 = 2''$

Showing $3 \cdot 2 = 6$:

1: $3 \cdot (1')$	Definition of $2 = 1'$
2: $(3 \cdot 1) + 3$	Axiom 14
3: $(3 \cdot 0') + 3$	Definition of $1 = 0'$
4: $(3 \cdot 0 + 3) + 3$	Axiom 14
5: $(0 + 3) + 3$	Axiom 13
6: $(3) + 3$	Addition
7: 6	Addition

#2 Formalize and justify. I'll start off with an example:

Proposition: If $a \le b$ and $c \le d$, then $ac \le bd$.

Formal Statement: $\forall a\, \forall b\, \forall c\, \forall d\, ((\exists k\, (b = a + k) \wedge \exists \ell\, (d = c + \ell)) \to \exists m\, (cd = bd + m))$

Fairly-Formal Proof:

1: $a \le b$	Given
2: $c \le d$	Given
3: There exists k such that $b = a + k$	Definition of inequality using 1
4: There exists ℓ such that $d = c + \ell$	Definition of inequality using 2
5: $bd = (a + k)(c + \ell)$	Equal values give equal answers
6: $(a + k)(c + \ell) = a(c + \ell) + k(c + \ell)$	Distributive law
7: $a(c + \ell) + k(c + \ell) = ac + a\ell + kc + k\ell$	Distributive laws
8: $bd = ac + (a\ell + kc + k\ell)$	Transitivity of equality (and associativity) using 5–7
9: $ac \le bd$	Definition of inequality using 8

Now fill in the formal statement and give justifications for the steps in my "Fairly-Formal" proof. I won't be too picky about your justifications. Use proper names where we have them.

Proposition: If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

Formal Statement: ???

Fairly-Formal Proof:

1: $a \mid b$

2: $a \mid c$

3: There exists k such that $b = a \cdot k$

4: There exists ℓ such that $c = a \cdot \ell$

5: $b + c = ak + a\ell$

6: $ak + a\ell = a(k + \ell)$

7: $b + c = a(k + \ell)$

8: $a \mid (b + c)$

#3 Prove that an odd number times an odd number is an odd number.

#4 Prove that $a \le b$ implies that $a + x \le b + x$.

#5 Prove that $a \,\Big|\, b$ and $b \,\Big|\, c$ implies that $a \,\Big|\, c$.

#6 Use mathematical induction to prove that the sum of the first n odd numbers is n^2.

In particular, show that $\sum_{i=1}^{n}(2i-1) = n^2$.

#7 When $n \ge ???$, we have $n^2 < n!$ where $n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$ (i.e., the factorial function). Find the smallest value for ??? that makes this true. Then prove this inequality using mathematical induction.

Hint: When n is positive, $1 \le n \le n^2$.