#1 In class we showed $2 + 2 = 4$ and $3 \cdot 2 = 6$ more-or-less using Axioms 11–14 on our E and PA handout.

(a) Show $4 + 3 = 7$.
(b) Show $5 \cdot 3 = 15$.

Use the same formatting and level of detail as my “proofs” shown below. You may “bury the details” of addition when working through the multiplication problem.

Showing $2 + 2 = 4$:
1: $2 + (1')$ Definition of 2 = 1'
2: $(2 + 1)'$ Axiom 12
3: $(2 + 0')'$ Definition of 1 = 0'
4: $((2 + 0)'')'$ Axiom 12
5: $(2)'$ Axiom 11
6: 4 Definition of 4 = 2'

Showing $3 \cdot 2 = 6$:
1: $3 \cdot (1')$ Definition of 1'
2: $(3 \cdot 1) + 3$ Axiom 14
3: $(3 \cdot 0') + 3$ Definition of 1 = 0'
4: $(3 \cdot 0 + 3) + 3$ Axiom 14
5: $(0 + 3) + 3$ Axiom 13
6: $3$ + $3$ Addition
7: 6 Addition

#2 Formalize and justify. I’ll start off with an example:

**Proposition:** If $a \leq b$ and $c \leq d$, then $ac \leq bd$.

**Formal Statement:** $\forall a \forall b \forall c \forall d \,(\exists k \,(b = a + k) \land \exists \ell \,(d = c + \ell)) \rightarrow \exists m \,(cd = bd + m))$

**Fairly-Formal Proof:**

1: $a \leq b$ Given
2: $c \leq d$ Given
3: There exists $k$ such that $b = a + k$ Definition of inequality using 1
4: There exists $\ell$ such that $d = c + \ell$ Definition of inequality using 2
5: $bd = (a + k)c + k \ell$ Equal values give equal answers
6: $(a + k)(c + \ell) = a(c + \ell) + k(c + \ell)$ Distributive law
7: $a(c + \ell) + k(c + \ell) = ac + a\ell + kc + k\ell$ Distributive laws
8: $bd = ac + (a\ell + kc + k\ell)$ Transitivity of equality (and associativity) using 5–7
9: $ac \leq bd$ Definition of inequality using 8

Now fill in the formal statement and give justifications for the steps in my “Fairly-Formal” proof. I won’t be too picky about your justifications. Use proper names where we have them.

**Proposition:** If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

**Formal Statement:** ???

**Fairly-Formal Proof:**

1: $a \\ b$ Given
2: $a \\ c$ Given
3: There exists $k$ such that $b = a \cdot k$ Definition of inequality using 1
4: There exists $\ell$ such that $c = a \cdot \ell$ Definition of inequality using 2
5: $b + c = ak + a\ell$
6: $ak + a\ell = a(k + \ell)$ Distributive laws
7: $b + c = a(k + \ell)$ Transitivity of equality (and associativity) using 5–7
8: $a \mid (b + c)$ Definition of inequality using 8
#3 Prove that an odd number times an odd number is an odd number.

#4 Prove that \( a \leq b \) implies that \( a + x \leq b + x \).

#5 Prove that \( a \mid b \) and \( b \mid c \) implies that \( a \mid c \).

#6 Use mathematical induction to prove that the sum of the first \( n \) odd numbers is \( n^2 \).

In particular, show that \( \sum_{i=1}^{n} (2i - 1) = n^2 \).

#7 When \( n \geq ??? \), we have \( n^2 < n! \) where \( n! = n(n-1) \cdots 3 \cdot 2 \cdot 1 \) (i.e., the factorial function). Find the smallest value for ??? that makes this true. Then prove this inequality using mathematical induction.

*Hint:* When \( n \) is positive, \( 1 \leq n \leq n^2 \).