Math 2110 # Homework #8 **Due: Fri., Oct. 23^{rd}, 2020**

Handwritten solutions are fine. ☺

Note: In this homework we will let $\emptyset = \{\}$ denote the empty set. Also, recall that $A - B = \{a \in A \mid a \notin B\}$ is the *relative complement* of B in A (or the *set difference* between A and B).

#1 Short proof. Let $\mathbb{Z}$ be the integers, $\mathbb{E}$ the even integers, and $\mathbb{O}$ the odd integers.

Use set builder notation to define $\mathbb{E}$ and $\mathbb{O}$. Then prove that $\mathbb{Z} = \mathbb{E} \cup \mathbb{O}$. Then explain why $\mathbb{E} \cap \mathbb{O} = \emptyset$.

#2 Let A and B be sets. Prove that $A = (A - B) \cup (A \cap B)$.

Include a Venn diagram picture to help illustrate why this is true (this picture is *not* sufficient as a proof).

#3 Let $A = \{8n \mid n \in \mathbb{Z}\}$, $B = \{8n + 4 \in \mathbb{Z} \mid n \in \mathbb{Z}\}$, and $C = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 4\}$. Prove that $A \cup B = C$.

#4 Prove that $\bigcup\{[n, n+1] \mid n \in \mathbb{Z}\} = \mathbb{R}$. Alternate notation: $\bigcup_{n \in \mathbb{Z}} [n, n+1] = \mathbb{R}$.

Notes: The set $[n, n+1] = \{x \in \mathbb{R} \mid n \leq x \leq n+1\}$ is the closed interval from n to $n+1$. Also, you probably will want to recall that $\lfloor r \rfloor \leq r < \lfloor r \rfloor + 1$ for any real number r where the floor function of r, $\lfloor r \rfloor$, just rounds down to the nearest integer.

#5 Prove that given a set A with m elements and a set B with n elements that $A \times B$ has $m \cdot n$ elements.

Hint/Recommendation: Use induction on the size of B.

An an example, I will carefully prove, given a set X with n elements, we have that $\mathcal{P}(X)$ has 2^n elements.

$\boxed{\textbf{Prop:}}$ Let X be a set of n elements (for some natural number n), then $\mathcal{P}(X)$, the power set of X, has 2^n elements.

Proof: We will proceed by induction on the size of our sets.

Base case: Let X be a set of 0 elements. Then $X = \emptyset$ and so $\mathcal{P}(X) = \mathcal{P}(\emptyset) = \{\emptyset\}$. Thus $\mathcal{P}(X)$ has $1 = 2^0$ elements. Therefore, the result holds for sets of size 0.

Inductive step: Fix some natural number n and assume that if X has n elements, we have that $\mathcal{P}(X)$ has 2^n elements (this is our inductive hypothesis).

Consider some set X with $n+1$ elements. In particular, suppose $X = \{x_1, x_2, \ldots, x_n, x_{n+1}\}$.

- Case 1: Consider subsets of X, $A \subseteq X$, such that $x_{n+1} \notin A$. These subsets are precisely the elements of $\mathcal{P}(\{x_1, \ldots, x_n\})$. By induction, we have that $\mathcal{P}(\{x_1, \ldots, x_n\})$ has size 2^n since $\{x_1, \ldots, x_n\}$ has size n. Thus there are exactly 2^n subsets of X that do not include the element x_{n+1}.
- Case 2: Consider subsets of X, $A \subseteq X$, such that $x_{n+1} \in A$. Given any such subset A, we have that $A - \{x_{n+1}\}$ belongs to $\mathcal{P}(\{x_1, \ldots, x_n\})$. Conversely each element $B \in \mathcal{P}(\{x_1, \ldots, x_n\})$ yields a subset $A = B \cup \{x_{n+1}\} \subseteq X$ where $x_{n+1} \in A$. Again, by induction, we know $\mathcal{P}(\{x_1, \ldots, x_n\})$ has exactly 2^n elements. Since the subsets of X that contain x_{n+1} are in a one-to-one correspondence with these sets, we must have exactly 2^n subsets of X that include the element x_{n+1}.

Now since every subset of X either includes x_{n+1} or not, these cases account for all subsets of X. Also, there is no overlap between these cases (a set cannot both contain x_{n+1} and simultaneously not contain x_{n+1}). Thus we must have a total of $2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$ subsets of X. In other words, the size of $\mathcal{P}(X)$ is 2^{n+1}. Therefore, if our statement is true for n, it must also be true for $n+1$.

Therefore, by induction, the result holds for all natural numbers n. ■