Math 2110 — # Homework #9 — **Due: Fri., Oct. 30^{th}, 2020**

Handwritten solutions are fine. ☺

Notational Notes: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ are integers, rational numbers, and real numbers (respectively). Also, $\mathbb{R}_{\geq 0} = [0, \infty)$ (i.e., non-negative reals).

#1 The following are faulty "definitions" of functions. Explain why each fails to be a function.

(a) Let $f : \mathbb{Q} \to \mathbb{Z}$ be defined by $f(a/b) = ab$ where $a, b \in \mathbb{Z}$ and $b \neq 0$ (i.e., a/b is a rational number).

(b) Let $g : \mathbb{Z} \to \mathbb{R}$ be defined by $g(n) = \sqrt{n}$.

#2 For each of the following functions, determine if the function is one-to-one, onto, both, or neither. Prove your answers.

(a) Let $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be defined by $f(x) = x^2 + 1$.

(b) Let $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $g(n) = \begin{cases} \dfrac{n}{2} & n \text{ is even} \\ 3 - n & n \text{ is odd} \end{cases}$

(c) Let $h : \mathbb{Z} \to \mathbb{Z}$ be defined by $h(n) = \begin{cases} n+1 & n \text{ is even} \\ 2n & n \text{ is odd} \end{cases}$

#3 Let $f : A \to B$ and $g : B \to C$ be surjective (i.e., onto) functions. Show that $g \circ f : A \to C$ is surjective.

#4 Let $f : A \to B$ be a function and let $S_1, S_2 \subseteq A$ and $T_1, T_2 \subseteq B$. Prove the following:

(a) $f(S_1) - f(S_2) \subseteq f(S_1 - S_2)$

(b) $f^{-1}(T_1 \cup T_2) = f^{-1}(T_1) \cup f^{-1}(T_2)$

(c) If $T_1 \subseteq T_2$, then $f^{-1}(T_1) \subseteq f^{-1}(T_2)$.

Keep in mind that for a function $f : A \to B$ where $S \subseteq A$ and $T \subseteq B$, we have the following equivalences (straight from the definitions):

- $x \in f(S)$ if and only if there exists some $y \in S$ such that $f(y) = x$.
- $x \in f^{-1}(T)$ if and only if $f(x) \in T$.

Here is a sample proof involving images and preimages (aka inverse images) of sets.

$\boxed{\textbf{Prop:}}$ Let $f : A \to B$ be a function.

- If $C \subseteq B$, then $f(f^{-1}(C)) \subseteq C$.
- f is onto if and only if for every $C \subseteq B$ we have $f(f^{-1}(C)) = C$.

Proof: For the first item, suppose $C \subseteq B$ and suppose $x \in f(f^{-1}(C))$. Then by definition, there exists some $y \in f^{-1}(C)$ such that $f(y) = x$. Again, by definition, $y \in f^{-1}(C)$ means $f(y) \in C$. Therefore, $x = f(y) \in C$. Thus $f(f^{-1}(C)) \subseteq C$.

For the second item, first suppose f is onto and let $C \subseteq B$. We already know that $f(f^{-1}(C)) \subseteq C$. So suppose $x \in C$. Then since f is onto, there exists some $y \in A$ such that $f(y) = x$. But then by definition $y \in f^{-1}(C)$ since $f(y) = x \in C$. Therefore, $f(y) \in f(f^{-1}(C))$ since $y \in f^{-1}(C)$ and so $x = f(y) \in f(f^{-1}(C))$. Thus we have shown $C \subseteq f(f^{-1}(C))$ and so $f(f^{-1}(C)) = C$.

Conversely, suppose that $f(f^{-1}(C) = C$ for all $C \subseteq B$. Let $y \in B$. Then since $\{y\} \subseteq B$, we have $f(f^{-1}(\{y\})) = \{y\}$. Thus, since $y \in \{y\} = f(f^{-1}(\{y\}))$, by definition there exists some $x \in f^{-1}(\{y\})$ such that $f(x) = y$. Again, by definition $f(x) \in \{y\}$ since $x \in f^{-1}(\{y\})$. So $f(x) = y$. Therefore, f is onto. ■