Proof Systems L and K Summary Sheets

Axioms:

Axiom 1: $A \to (B \to A)$

Axiom 2: $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Axiom 3: $(\neg B \to \neg A) \to ((\neg B \to A) \to B)$

Axiom 4: $(\forall x A(x)) \to A(t)$, provided that t is free for x in $A(x)$.

Axiom 5: $\forall x(A \to B) \to (A \to \forall x B)$, provided that x does not occur free in A.

Rules of Inference:

Modus Ponens (MP): From A and $A \to B$ deduce B.

Generalization (GEN): From A, deduce $\forall x A$.

Abbreviations:

Existential Quantifier: $\exists x A$ abbreviates $\neg \forall x \neg A$

Conjunction: $A \wedge B$ abbreviates $\neg(A \to \neg B)$

Disjunction: $A \vee B$ abbreviates $(\neg A) \to B$

Double Implication: $A \leftrightarrow B$ abbreviates $\neg((A \to B) \to \neg(B \to A))$
Note: When $A \leftrightarrow B$ and $(A \to B) \wedge (B \to A)$ are unabbreviated, they are identical.

Theorems in L:

L1: $\vdash_L A \to A$

L2: $\vdash_L (\neg B \to B) \to B$

L3: $A \to (B \to C),\ A \to B \vdash_L A \to C$

L4: $A \to ((B \to A) \to C) \vdash_L A \to C$

L5: $B \vdash_L A \to B$

L6: $A \to (B \to C),\ B \vdash_L A \to C$

L7: $A \to (B \to C) \vdash_L B \to (A \to C)$

L8: $A \to B,\ B \to C \vdash_L A \to C$

L9: $P \to R \vdash_L P \to (Q \to R)$

L10: $\vdash_L (\neg B \to \neg A) \to (A \to B)$

L11: $\vdash_L \neg\neg B \to B$

L12: $\vdash_L B \to \neg\neg B$

L13: $\vdash_L (A \to (B \to C)) \to (B \to (A \to C))$

L14: $\vdash_L A \to ((A \to B) \to B)$

L15: $\vdash_L \neg A \to (A \to B)$

L16: $\vdash_L (A \to B) \to (\neg B \to \neg A)$

L17: $\vdash_L A \to (\neg B \to \neg(A \to B))$

L18: $A,\ B \vdash_L A \wedge B$

L19: $A \wedge B \vdash_L A$

L20: $A \wedge B \vdash_L B$

L21: $A \vdash_L A \vee B$

L22: $B \vdash_L A \vee B$

L23: $A \leftrightarrow B \vdash_L A \to B$

L24: $A \leftrightarrow B \vdash_L B \to A$

L25: $A \to B, B \to A \vdash_L A \leftrightarrow B$

Shortcuts:

Lemmas: You may write an instance of a previously proved (lowed numbered) theorem as a line, if the instances of the "givens" of that theorem appear as earlier lines.

Rule T: Any instance of a tautology may be inserted as a line in a predicate calculus proof.

Deduction Theorem: If there is a proof of $A \vdash B$ with no applications of generalization to any variables that occur free in A, then there is a proof of $\vdash A \to B$.

Add $\exists x$ Rule: If $A(t)$ is the result of replacing every free occurrence of x in $A(x)$ with t, and t is free for x in $A(x)$, then from $A(t)$ we may deduce $\exists x A(x)$.

Rule C: If $\exists x A(x)$ is a previous line in a proof, we may write $A(\underline{c})$ as a line, provided that the following two conditions hold.

1. $\underline{c}$ is a new constant symbol.
2. If some variable (say y) appears free in the formula $\exists x A(x)$, then GEN is never applied to y in the proof.

Theorems in K:

K1: $\vdash \forall x(A(x) \to (B(x) \to A(x)))$.

K2: $\forall x \forall y A(x,y) \;\vdash\; \forall y \forall x A(x,y)$.

K3: $A(x) \wedge B(x) \;\vdash\; A(x)$.

K4: $\vdash \forall x \forall y A(x,y) \to \forall y \forall x A(x,y)$

K5: $\vdash \forall x(A(x) \wedge B(x)) \to \forall x A(x)$

K6: $\vdash [\forall x A(x)] \to \forall x(A(x) \vee B(x))$

K7: $\vdash \forall x(A(x) \to B(x)) \to (\forall x A(x) \to \forall x B(x))$

K8: $\vdash \forall x B(x) \to \forall x(A(x) \to B(x))$

K9: $\vdash \forall x \forall y A(x,y) \to \forall y \forall x A(y,x)$

K10: $\vdash \forall x(A(x) \vee B(x)) \to (\forall x \neg A(x) \to \forall x B(x))$

K11: $A(x) \vdash \exists x A(x)$

K12: $A(y,y) \vdash \forall y \exists x A(x,y)$

K13: $\vdash \forall x A(x) \to \exists x A(x)$

K14: $\vdash \forall y A(y) \to \exists x A(x)$

K15: $\vdash \forall x(A(x) \vee B(x)) \to [\forall x A(x) \vee \exists x B(x)]$

K16: $\neg \exists x A(x) \vdash \forall x \neg A(x)$

K17: $\forall x \neg A(x) \vdash \neg \exists x A(x)$

K18: $\exists x \neg A(x) \vdash \neg \forall x A(x)$

K19: $\neg \forall x A(x) \vdash \exists x \neg A(x)$

K20: $\exists x(A(x) \wedge B(x)) \vdash \exists x A(x)$

K21: $\vdash \exists x A(x) \to \exists x(A(x) \vee B(x))$

K22: $\vdash \exists x \forall y A(x,y) \to \forall y \exists x A(x,y)$

K23: $\vdash \exists x(A(x) \to B(x)) \to (\forall x A(x) \to \exists x B(x))$

K24: $\vdash \exists x B(x) \to \exists x(A(x) \to B(x))$

K25: $\vdash \neg \forall x A(x) \to \exists x(A(x) \to B(x))$

K26: $\vdash (\forall x A(x) \to \exists x B(x)) \to \exists x(A(x) \to B(x))$

K27: $\vdash \forall x A(x,x) \to \forall x \exists y A(x,y)$

K28: $\vdash \forall y \exists x(\neg A(y,x) \vee A(y,y))$

K29: $\vdash \exists x(A(x) \vee B(x)) \to (\exists x A(x) \vee \exists x B(x))$

K30: $\vdash (\exists x A(x) \vee \exists x B(x)) \to \exists x(A(x) \vee B(x))$.

K31: $\vdash \exists x(A(x) \wedge B(x)) \to (\exists x A(x) \wedge \exists x B(x))$

K32: $\vdash (\forall x A(x) \wedge \exists x B(x)) \to \exists x(A(x) \wedge B(x))$

K33: $\vdash (\forall x A(x) \vee \forall x B(x)) \to \forall x(A(x) \vee B(x))$

K34: $\vdash \forall x(A(x) \wedge B(x)) \to (\forall x A(x) \wedge \forall x B(x))$

K35: $\vdash (\forall x A(x) \wedge \forall x B(x)) \to \forall x(A(x) \wedge B(x))$

K36: $\vdash (\exists x A(x) \to \forall x B(x)) \to \forall x(A(x) \to B(x))$