Affine Lie Algebras, Vertex Operator Algebras, and Multisum Identities

Technical Slides

William J. Cook

Rutgers University

Department of Mathematics

New Brunswick, NJ

wjcook@math.rutgers.edu

joint work with

Kailash Misra and Haisheng Li

Outline

- highest-weight integrable modules for affine Lie algebras and their VOA-module structure
- VOA-module isomorphisms and recurrence relations
- a character formula for $L(\Lambda_0)$
- principal characters and multisum identities

Notation:

- Let $\mathfrak{g}$ be a simple Lie algebra (over $\mathbb{C}$) of rank ℓ.

Notation:

- Let $\mathfrak{g}$ be a simple Lie algebra (over $\mathbb{C}$) of rank ℓ.
- Let $C = (a_{ij})_{1 \le i,j \le \ell}$ be the Cartan matrix of $\mathfrak{g}$.

Notation:

- Let $\mathfrak{g}$ be a simple Lie algebra (over $\mathbb{C}$) of rank ℓ.
- Let $C = (a_{ij})_{1 \leq i,j \leq \ell}$ be the Cartan matrix of $\mathfrak{g}$.
- Fix a set of simple roots. Define the root lattice to be the $\mathbb{Z}$-span of these roots:

$$Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \ldots + \mathbb{Z}\alpha_\ell.$$

Notation:

- Let $\mathfrak{g}$ be a simple Lie algebra (over $\mathbb{C}$) of rank ℓ.
- Let $C = (a_{ij})_{1 \leq i,j \leq \ell}$ be the Cartan matrix of $\mathfrak{g}$.
- Fix a set of simple roots. Define the root lattice to be the $\mathbb{Z}$-span of these roots:

$$Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \ldots + \mathbb{Z}\alpha_\ell.$$

- Define the weight lattice to be the $\mathbb{Z}$-span of the fundamental weights:

$$P = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2 + \ldots + \mathbb{Z}\lambda_\ell.$$

Notation:

- Fix Chevalley generators $\{E_i, F_i, H_i \,|\, 1 \le i \le \ell\}$.

Notation:

- Fix Chevalley generators $\{E_i, F_i, H_i \mid 1 \le i \le \ell\}$.
- The H_i $(1 \le i \le \ell)$ span a Cartan subalgebra (CSA) denoted by $\mathfrak{h}$. (Note: $\dim(\mathfrak{h}) = \ell$.)

Notation:

- Fix Chevalley generators $\{E_i, F_i, H_i \mid 1 \le i \le \ell\}$.
- The H_i $(1 \le i \le \ell)$ span a Cartan subalgebra (CSA) denoted by $\mathfrak{h}$. (Note: $\dim(\mathfrak{h}) = \ell$.)
- Let $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ be the normalized nondegenerate symmetric invariant bilinear form.

Notation:

- Fix Chevalley generators $\{E_i, F_i, H_i \,|\, 1 \leq i \leq \ell\}$.
- The H_i $(1 \leq i \leq \ell)$ span a Cartan subalgebra (CSA) denoted by $\mathfrak{h}$. (Note: $\dim(\mathfrak{h}) = \ell$.)
- Let $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ be the normalized nondegenerate symmetric invariant bilinear form.
- We have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where Δ is the set of roots.

Notation:

- By definition, $\{H_1, \ldots, H_\ell\}$ and $\{\lambda_1, \ldots, \lambda_\ell\}$ are dual bases (i.e. $\lambda_i(H_j) = \delta_{i,j}$).

Notation:

- By definition, $\{H_1, ..., H_\ell\}$ and $\{\lambda_1, ..., \lambda_\ell\}$ are dual bases (i.e. $\lambda_i(H_j) = \delta_{i,j}$).
- Define the fundamental coweights, $\{H^{(1)}, ..., H^{(\ell)}\} \subseteq \mathfrak{h}$, to be the basis dual to $\{\alpha_1, ..., \alpha_\ell\}$.

Notation:

- By definition, $\{H_1, \ldots, H_\ell\}$ and $\{\lambda_1, \ldots, \lambda_\ell\}$ are dual bases (i.e. $\lambda_i(H_j) = \delta_{i,j}$).
- Define the fundamental coweights, $\{H^{(1)}, \ldots, H^{(\ell)}\} \subseteq \mathfrak{h}$, to be the basis dual to $\{\alpha_1, \ldots, \alpha_\ell\}$.
- Define the coroot lattice:

$$Q^\vee = \mathbb{Z}H_1 + \mathbb{Z}H_2 + \ldots + \mathbb{Z}H_\ell.$$

Notation:

- By definition, $\{H_1, \ldots, H_\ell\}$ and $\{\lambda_1, \ldots, \lambda_\ell\}$ are dual bases (i.e. $\lambda_i(H_j) = \delta_{i,j}$).
- Define the fundamental coweights, $\{H^{(1)}, \ldots, H^{(\ell)}\} \subseteq \mathfrak{h}$, to be the basis dual to $\{\alpha_1, \ldots, \alpha_\ell\}$.
- Define the coroot lattice:

$$Q^\vee = \mathbb{Z}H_1 + \mathbb{Z}H_2 + \ldots + \mathbb{Z}H_\ell.$$

- Define the coweight lattice:

$$P^\vee = \mathbb{Z}H^{(1)} + \mathbb{Z}H^{(2)} + \ldots + \mathbb{Z}H^{(\ell)}.$$

Notation:

- Let $\rho^\vee = \sum_{i=1}^{\ell} H^{(i)}$.

Notation:

- Let $\rho^\vee = \sum_{i=1}^{\ell} H^{(i)}$.
- We have that $\langle H_i, H^{(j)} \rangle = \frac{2}{\langle \alpha_i, \alpha_i \rangle}$.

Notation:

- Let $\rho^\vee = \sum_{i=1}^{\ell} H^{(i)}$.
- We have that $\langle H_i, H^{(j)} \rangle = \frac{2}{\langle \alpha_i, \alpha_i \rangle}$.
- The coroots and coweights are related as follows:

$$H_j = \sum_{i=1}^{\ell} a_{ij} H^{(i)}$$

where $C = (a_{ij})$ is the Cartan matrix of $\mathfrak{g}$.

The Affinization of $\mathfrak{g}$

Define $\hat{\mathfrak{g}}$ as follows:

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

where c is central and

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b\rangle \delta_{m+n,0} c$$

for every $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$.
Extend the map $d = 1 \otimes t\frac{\mathrm{d}}{\mathrm{d}t}$ to all of $\hat{\mathfrak{g}}$ by letting $d(c) = 0$. Let $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \rtimes \mathbb{C}d$ be the semi-direct product Lie algebra.
$\tilde{\mathfrak{g}}$ is the (untwisted) affine Lie algebra associated with $\mathfrak{g}$.

Chevalley Generators for $\tilde{\mathfrak{g}}$

- Let $\theta = \sum_{i=1}^{\ell} a_i \alpha_i$ be the highest (long) root of $\mathfrak{g}$.

Chevalley Generators for $\tilde{\mathfrak{g}}$

- Let $\theta = \sum_{i=1}^{\ell} a_i \alpha_i$ be the highest (long) root of $\mathfrak{g}$.
- Let E_θ and F_θ be elements of the root spaces $\mathfrak{g}_\theta$ and $\mathfrak{g}_{-\theta}$ respectively such that $\langle E_\theta, F_\theta \rangle = 1$. Set $H_\theta = [E_\theta, F_\theta]$.

Chevalley Generators for $\tilde{\mathfrak{g}}$

- Let $\theta = \sum_{i=1}^{\ell} a_i \alpha_i$ be the highest (long) root of $\mathfrak{g}$.
- Let E_θ and F_θ be elements of the root spaces $\mathfrak{g}_\theta$ and $\mathfrak{g}_{-\theta}$ respectively such that $\langle E_\theta, F_\theta \rangle = 1$. Set $H_\theta = [E_\theta, F_\theta]$.
- Let $e_i = E_i \otimes 1$, $f_i = F_i \otimes 1$, and $h_i = H_i \otimes 1$ for $1 \leq i \leq \ell$.
- Let $e_0 = F_\theta \otimes t$, $f_0 = E_\theta \otimes t^{-1}$, and $h_0 = [e_0, f_0]$.

Chevalley Generators for $\tilde{\mathfrak{g}}$

- Let $\theta = \sum_{i=1}^{\ell} a_i \alpha_i$ be the highest (long) root of $\mathfrak{g}$.
- Let E_θ and F_θ be elements of the root spaces $\mathfrak{g}_\theta$ and $\mathfrak{g}_{-\theta}$ respectively such that $\langle E_\theta, F_\theta \rangle = 1$. Set $H_\theta = [E_\theta, F_\theta]$.
- Let $e_i = E_i \otimes 1$, $f_i = F_i \otimes 1$, and $h_i = H_i \otimes 1$ for $1 \leq i \leq \ell$.
- Let $e_0 = F_\theta \otimes t$, $f_0 = E_\theta \otimes t^{-1}$, and $h_0 = [e_0, f_0]$.

Then, $\{e_i, f_i, h_i \,|\, 0 \leq i \leq \ell\}$ are Chevalley generators for $\tilde{\mathfrak{g}}$. Also, $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ is our CSA.

The Homogeneous Gradation

The Homogeneous Gradation

Thus we have the following decomposition of $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}} = \coprod_{n\in\mathbb{Z}} \hat{\mathfrak{g}}_{(n)} \qquad \text{where} \qquad \hat{\mathfrak{g}}_{(n)} = \begin{cases} \mathfrak{g} \oplus \mathbb{C}c & n = 0 \\ \mathfrak{g} \otimes t^{-n} & n \neq 0 \end{cases}$$

We have the following graded subalgebras of $\hat{\mathfrak{g}}$:

$$\hat{\mathfrak{g}}_{(\pm)} = \coprod_{n>0} \hat{\mathfrak{g}}_{(\pm n)} = \coprod_{n>0} \mathfrak{g} \otimes t^{\mp n}$$

and

$$\hat{\mathfrak{g}}_{(\leq 0)} = \coprod_{n\leq 0} \hat{\mathfrak{g}}_{(n)} = \hat{\mathfrak{g}}_{(-)} \oplus \hat{\mathfrak{g}}_{(0)} = \hat{\mathfrak{g}}_{(-)} \oplus \mathfrak{g} \oplus \mathbb{C}c.$$

The Principal Gradation

The second gradation we shall consider is the principal gradation.

The principal gradation is given by the derivation d_P which we define on the Chevalley generators of $\hat{\mathfrak{g}}$ and extend using the Leibnitz rule. For $0 \leq i \leq \ell$, let

- $d_P(e_i) = e_i$,
- $d_P(f_i) = -f_i$, and
- $d_P(h_i) = 0$.

The Principal Gradation

The second gradation we shall consider is the principal gradation.

The principal gradation is given by the derivation d_P which we define on the Chevalley generators of $\hat{\mathfrak{g}}$ and extend using the Leibnitz rule. For $0 \leq i \leq \ell$, let

- $d_P(e_i) = e_i$,
- $d_P(f_i) = -f_i$, and
- $d_P(h_i) = 0$.

Lemma: On $\hat{\mathfrak{g}}$,

$$d_P = (\mathrm{ht}(\theta) + 1)d + \mathrm{ad}(\rho^\vee)$$

where θ is the highest long root of $\mathfrak{g}$.

From $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$

Given $\lambda \in \mathfrak{h}^*$ and $k \in \mathbb{C}$, we can define a linear functional $(k, \lambda) \in \tilde{\mathfrak{h}}^*$ by the following:

- For all $h \in \mathfrak{h}$, let $(k, \lambda)(h) = \lambda(h)$.
- On $\mathbb{C}c$, let $(k, \lambda)(c) = k$.
- On $\mathbb{C}d$, let $(k, \lambda)(d) = 0$.

From $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$

Given $\lambda \in \mathfrak{h}^*$ and $k \in \mathbb{C}$, we can define a linear functional $(k, \lambda) \in \tilde{\mathfrak{h}}^*$ by the following:

- For all $h \in \mathfrak{h}$, let $(k, \lambda)(h) = \lambda(h)$.
- On $\mathbb{C}c$, let $(k, \lambda)(c) = k$.
- On $\mathbb{C}d$, let $(k, \lambda)(d) = 0$.

Recall that λ_i ($1 \leq i \leq \ell$) are the fundamental weights of $\mathfrak{g}$. For convenience, let $\lambda_0 = 0$. Then, define $k\Lambda_i = (k, \lambda_i)$ for $0 \leq i \leq \ell$.

From $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$

Given $\lambda \in \mathfrak{h}^*$ and $k \in \mathbb{C}$, we can define a linear functional $(k, \lambda) \in \tilde{\mathfrak{h}}^*$ by the following:

- For all $h \in \mathfrak{h}$, let $(k, \lambda)(h) = \lambda(h)$.
- On $\mathbb{C}c$, let $(k, \lambda)(c) = k$.
- On $\mathbb{C}d$, let $(k, \lambda)(d) = 0$.

Recall that λ_i ($1 \leq i \leq \ell$) are the fundamental weights of $\mathfrak{g}$. For convenience, let $\lambda_0 = 0$. Then, define $k\Lambda_i = (k, \lambda_i)$ for $0 \leq i \leq \ell$.

$\Lambda_0 = (1, \lambda_0) = (1, 0)$, $\Lambda_1 = (1, \lambda_1)$, ..., $\Lambda_\ell = (1, \lambda_\ell)$ are the fundamental weights for $\tilde{\mathfrak{g}}$.

The Verma Module $V(k, \lambda)$

Fix a complex number $k \in \mathbb{C}$ and let U be the irreducible finite dimensional $\mathfrak{g}$-module with highest-weight $\lambda \in P_+$. Extend the $\mathfrak{g}$-module, U, to a $\hat{\mathfrak{g}}_{(\leq 0)}$-module U_k as follows:

The Verma Module $V(k, \lambda)$

Fix a complex number $k \in \mathbb{C}$ and let U be the irreducible finite dimensional $\mathfrak{g}$-module with highest-weight $\lambda \in P_+$. Extend the $\mathfrak{g}$-module, U, to a $\hat{\mathfrak{g}}_{(\leq 0)}$-module U_k as follows:

- Let $U_k = U$ (as a $\mathbb{C}$-vector space).

The Verma Module $V(k, \lambda)$

Fix a complex number $k \in \mathbb{C}$ and let U be the irreducible finite dimensional $\mathfrak{g}$-module with highest-weight $\lambda \in P_+$. Extend the $\mathfrak{g}$-module, U, to a $\hat{\mathfrak{g}}_{(\leq 0)}$-module U_k as follows:

- Let $U_k = U$ (as a $\mathbb{C}$-vector space).
- Let $\hat{\mathfrak{g}}_{(-)}$ act trivially on U_k.

The Verma Module $V(k, \lambda)$

Fix a complex number $k \in \mathbb{C}$ and let U be the irreducible finite dimensional $\mathfrak{g}$-module with highest-weight $\lambda \in P_+$. Extend the $\mathfrak{g}$-module, U, to a $\hat{\mathfrak{g}}_{(\leq 0)}$-module U_k as follows:

- Let $U_k = U$ (as a $\mathbb{C}$-vector space).
- Let $\hat{\mathfrak{g}}_{(-)}$ act trivially on U_k.
- Let c act as the scalar $k1_U$ (i.e. $c \cdot u = ku$ for all $u \in U$).

The Verma Module $V(k, \lambda)$

Fix a complex number $k \in \mathbb{C}$ and let U be the irreducible finite dimensional $\mathfrak{g}$-module with highest-weight $\lambda \in P_+$. Extend the $\mathfrak{g}$-module, U, to a $\hat{\mathfrak{g}}_{(\leq 0)}$-module U_k as follows:

- Let $U_k = U$ (as a $\mathbb{C}$-vector space).
- Let $\hat{\mathfrak{g}}_{(-)}$ act trivially on U_k.
- Let c act as the scalar $k1_U$ (i.e. $c \cdot u = ku$ for all $u \in U$).

Define the Verma module (of level k and highest-weight (k, λ)) as follows:

$$V(k, \lambda) = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{(\leq 0)})} U_k.$$

The Simple VOA $L(k\Lambda_0)$

Theorem: Suppose that $k \neq -h^\vee$ the dual Coxeter number. Then,

- $V(k\Lambda_0) = V(k, 0)$ is a vertex operator algebra,

The Simple VOA $L(k\Lambda_0)$

Theorem: Suppose that $k \neq -h^\vee$ the dual Coxeter number. Then,

- $V(k\Lambda_0) = V(k, 0)$ is a vertex operator algebra,
- N (the maximal proper submodule) is an ideal of $V(k\Lambda_0)$, and
- $L(k\Lambda_0)$ $(= V(k\Lambda_0)/N$ $)$ is a simple vertex operator algebra.

The Simple VOA $L(k\Lambda_0)$

Theorem: Suppose that $k \neq -h^\vee$ the dual Coxeter number. Then,

- $V(k\Lambda_0) = V(k, 0)$ is a vertex operator algebra,
- N (the maximal proper submodule) is an ideal of $V(k\Lambda_0)$, and
- $L(k\Lambda_0)$ $(= V(k\Lambda_0)/N)$ is a simple vertex operator algebra.

Theorem: If $k \neq -h^\vee$ and $\lambda \in P_+$, then $L(k, \lambda)$ $(= V(k, \lambda)/$(maximal submodule) is an irreducible $V(k\Lambda_0)$-module.

Li's Theorem

For all that follows, let $k \in \mathbb{Z}_{>0}$.

Li's Theorem

For all that follows, let $k \in \mathbb{Z}_{>0}$.

Let $H \in P^{\vee}$. Define

$$\Delta(H,x) = x^{H_0} \exp\left(\sum_{k=1}^{\infty} \frac{H_k}{-k}(-x)^{-k}\right).$$

Li's Theorem

For all that follows, let $k \in \mathbb{Z}_{>0}$.

Let $H \in P^\vee$. Define

$$\Delta(H, x) = x^{H_0} \exp\left(\sum_{k=1}^{\infty} \frac{H_k}{-k}(-x)^{-k}\right).$$

Theorem [Li]: If $L(k, \lambda)$ is an irreducible $L(k\Lambda_0)$-module and $H \in Q^\vee$, then

$$(L(k, \lambda), Y(\cdot, x))$$

is isomorphic to

$$(L(k, \lambda), Y(\Delta(H, x)\cdot, x))$$

as an $L(k\Lambda_0)$-module.

The Automorphism's Effect

Notation: For $v \in L(k\Lambda_0)$, let

$$Y(v,x) = \sum_{m\in\mathbb{Z}} v(m)x^{-m-1},$$

$$Y(\Delta(H,x)v,x) = \sum_{m\in\mathbb{Z}} v_{(H)}(m)x^{-m-1}$$

and

$$Y(\Delta(H,x)\omega,x) = \sum_{m\in\mathbb{Z}} L_{(H)}(m)x^{-m-2}.$$

The Automorphism's Effect

Lemma: For $1 \leq i, j \leq \ell$,

$$(H^{(j)})_{(H_i)}(0) = H^{(j)}(0) + \frac{2}{\langle \alpha_i, \alpha_i \rangle} \delta_{i,j} k$$

$$\begin{aligned} L_{(H_i)}(0) &= L(0) + H_i(0) + \frac{2}{\langle \alpha_i, \alpha_i \rangle} \delta_{i,j} k \\ &= L(0) + \sum_{j=1}^{\ell} a_{ji} H^{(j)}(0) + \frac{2}{\langle \alpha_i, \alpha_i \rangle} \delta_{i,j} k \end{aligned}$$

Characters

Definition: Let $L(k,\lambda)$ be an irreducible $L(k\Lambda_0)$-module.

The (full) character of $L(k,\lambda)$ is given by

$$\chi_{L(k,\lambda)}(x_1, x_2, ..., x_\ell; q) =$$

$$\mathrm{tr}_{L(k,\lambda)} x_1^{H^{(1)}(0)} x_2^{H^{(2)}(0)} ... x_\ell^{H^{(\ell)}(0)} q^{L(0)}.$$

The Main Theorem

Main Theorem: If $L(k, \lambda)$ is an irreducible $L(k\Lambda_0)$-module, then for each $1 \leq i \leq \ell$ we have that

$$\chi_{L(k,\lambda)}(x_1, x_2, ..., x_\ell; q) =$$

$$(x_i q)^{\frac{2k}{\langle \alpha_i, \alpha_i \rangle}} \chi_{L(k,\lambda)}(x_1 q^{a_{1i}}, x_2 q^{a_{2i}}, ..., x_\ell q^{a_{\ell i}}; q).$$

Recurrence Relations

Notation: Let

$$\mathbf{n} = (n_1, n_2, ..., n_\ell) \in \mathbb{Z}^\ell$$

and

$$\mathbf{x}^\mathbf{n} = x_1^{n_1} x_2^{n_2} ... x_\ell^{n_\ell}.$$

Recurrence Relations

Notation: Let

$$\mathbf{n} = (n_1, n_2, ..., n_\ell) \in \mathbb{Z}^\ell$$

and

$$\mathbf{x}^{\mathbf{n}} = x_1^{n_1} x_2^{n_2} ... x_\ell^{n_\ell}.$$

Define coefficients $A(\mathbf{n}; q) \in q^{-\mu}\mathbb{C}[[q]]$ as follows:

$$\chi_{L(k,\lambda)}(x_1, x_2, ..., x_\ell; q) = \sum_{\mathbf{n}\in\mathbb{Z}^\ell} A(\mathbf{n}; q)\mathbf{x}^{\mathbf{n}}$$

Note: $L(0) = -d + \mu$ for some $\mu \in \mathbb{C}$.

Recurrence Relations

Corollary: For $1 \le i \le \ell$ and $(n_1, ..., n_\ell) \in \mathbb{Z}^\ell$,

$$A(n_1, ..., n_\ell; q) =$$

$$A(n_1, ..., n_{i-1}, n_i - \frac{2k}{\langle \alpha_i, \alpha_i \rangle}, n_{i+1}, ..., n_\ell; q) q^{-\frac{2k}{\langle \alpha_i, \alpha_i \rangle} + \sum_{j=1}^{\ell} a_{ji} n_j}.$$

A Special Case

Let us specialize to level $k = 1$ and $\mathfrak{g}$ of (ADE)-type (simply laced). Then $\langle \alpha_i, \alpha_i \rangle = 2$ for all $1 \leq i \leq \ell$. So we have the following recurrence relations:

For $1 \leq i \leq \ell$ and $(n_1, n_2, ..., n_\ell) \in \mathbb{Z}^\ell$,

$$A(n_1, ..., n_\ell; q) =$$

$$A(n_1, ..., n_{i-1}, n_i - 1, n_{i+1}, ..., n_\ell; q) q^{-1+\sum_{j=1}^{\ell} a_{ji} n_j}$$

A Sum Formula

Using the lattice construction of $L(\Lambda_0) = V_Q$ we £nd that

$$
\begin{aligned}
A(0,...,0;q) &= \dim_q S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) \\
&= \prod_{j=1}^{\infty} (1-q^j)^{-\ell}.
\end{aligned}
$$

A Sum Formula

Using the lattice construction of $L(\Lambda_0) = V_Q$ we £nd that

$$A(0, \ldots, 0; q) = \dim_q S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}]) = \prod_{j=1}^{\infty} (1 - q^j)^{-\ell}.$$

Theorem: Let $\mathfrak{g}$ be a simple Lie algebra of (ADE)-type.

$$\chi_{L(\Lambda_0)}(\mathbf{x}; q) = \prod_{j=1}^{\infty} (1 - q^j)^{-\ell} \sum_{\mathbf{n} \in \mathbb{Z}^\ell} q^{\frac{1}{2}\mathbf{n}C\mathbf{n}^t} \mathbf{x}^{\mathbf{n}}$$

Characters

Recall that ($\mathrm{ht}(\theta) =$ height of the highest long root)

$$\begin{aligned} d_P &= (\mathrm{ht}(\theta)+1)d + ad(\rho^\vee) \\ &= (\mathrm{ht}(\theta)+1)(-L(0)+\mu) + ad(\rho^\vee) \end{aligned}$$

Characters

Recall that ($\mathrm{ht}(\theta) =$ height of the highest long root)

$$\begin{aligned} d_P &= (\mathrm{ht}(\theta)+1)d + ad(\rho^\vee) \\ &= (\mathrm{ht}(\theta)+1)(-L(0)+\mu) + ad(\rho^\vee) \end{aligned}$$

Definition: Let $L(k,\lambda)$ be an irreducible $L(k\Lambda_0)$-module.

The principal character of $L(k,\lambda)$ is given by

$$\begin{aligned} \chi^P_{L(k,\lambda)}(q) &= \mathrm{tr}_{L(k,\lambda)} q^{-d_P} \\ &= q^{-(\mathrm{ht}(\theta)+1)\mu} \chi_{L(k,\lambda)}(q^{-1}, \ldots q^{-1}; q^{\mathrm{ht}(\theta)+1}). \end{aligned}$$

The Principal Character

Corollary: Let $\mathfrak{g}$ be of (ADE)-type (i.e. simply laced) and $k = 1$. Then,

$$\chi^P_{L(\Lambda_0)}(q) =$$

$$\prod_{j\geq 1}(1 - q^{(\mathrm{ht}(\theta)+1)j})^{-\ell} \sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{\mathrm{ht}(\theta)+1}{2}\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{\ell} n_i}$$

The Principal Character

Corollary: Let $\mathfrak{g}$ be of (ADE)-type (i.e. simply laced) and $k = 1$. Then,

$$\chi^P_{L(\Lambda_0)}(q) =$$

$$\prod_{j \geq 1} (1 - q^{(\mathrm{ht}(\theta)+1)j})^{-\ell} \sum_{\mathbf{n} \in \mathbb{Z}^\ell} q^{\frac{\mathrm{ht}(\theta)+1}{2} \mathbf{n} C \mathbf{n}^t - \sum_{i=1}^{\ell} n_i}$$

Thus we have obtained a multisum formula for the principal character. Now using the previously known formulas, we obtain families of multisum identities...

Type A_ℓ

Theorem: If $C = (a_{ij})$ is the Cartan matrix of type A_ℓ ($\ell \geq 1$),

$$\prod_{j\geq 1} \frac{(1-q^{(\ell+1)j})^{(\ell+1)}}{(1-q^j)} = \sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{\frac{\ell+1}{2}\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^{\ell} n_i}.$$

Example: Type A_1

Example: For type A and rank $\ell = 1$, $C = (2)$. So we have,

$$\prod_{j \geq 1} \frac{(1 - q^{2j})^2}{(1 - q^j)} = \sum_{n \in \mathbb{Z}} q^{2n^2 - n}.$$

This is a well known formula credited to Gauss.

Example: Type A_2

Example: For type A and rank $\ell = 2$,

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

So we have,

$$\prod_{j\geq 1} \frac{(1-q^{3j})^3}{(1-q^j)} = \sum_{n_1,n_2\in\mathbb{Z}} q^{3(n_1^2-n_1n_2+n_2^2)-(n_1+n_2)}.$$

Type D_ℓ

Theorem: If $C = (a_{ij})$ is the Cartan matrix of type D_ℓ ($\ell \geq 4$),

$$\prod_{j\geq 1} \frac{(1-q^{2(\ell-1)j})^\ell}{(1-q^{2j-1})(1-q^{(\ell-1)(2j-1)})}$$

$$=$$

$$\sum_{\mathbf{n}\in\mathbb{Z}^\ell} q^{(\ell-1)\mathbf{n}C\mathbf{n}^t-\sum_{i=1}^{\ell} n_i}.$$

Example: Type D_4

Example: For type D and rank $\ell = 4$,

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

Example: Type D_4

Example: For type D and rank $\ell = 4$,

So we have,

$$\prod_{j\geq 1} \frac{(1-q^{6j})^4}{(1-q^{(2j-1)})(1-q^{(6j-3)})}$$

$$=$$

$$\sum_{n_1,\ldots,n_4\in\mathbb{Z}} q^{6(n_1^2+n_2^2+n_3^2+n_4^2-n_1n_2-n_2n_3-n_2n_4)-(n_1+n_2+n_3+n_4)}.$$

Type E_6

Theorem: If $C = (a_{ij})$ is the Cartan matrix of type E_6, then

$$\varphi(q^{12})^6 \prod_{j\equiv\pm1,\pm4,\pm5 \,(\mathrm{mod} 12)} (1-q^j)^{-1}$$

$$=$$

$$\sum_{\mathbf{n}\in\mathbb{Z}^6} q^{6\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^6 n_i}$$

where $\varphi(q) = \prod_{j\geq1}(1-q^j)$ is the Euler product function.

Type E_7

Theorem: If $C = (a_{ij})$ is the Cartan matrix of type E_7, then

$$\varphi(q^{18})^7 \prod_{j\equiv \pm 1, \pm 5, \pm 7, 9 \pmod{18}} (1-q^j)^{-1}$$

$$=$$

$$\sum_{\mathbf{n}\in\mathbb{Z}^7} q^{9\mathbf{n}C\mathbf{n}^t - \sum_{i=1}^7 n_i}$$

where $\varphi(q) = \prod_{j\geq 1}(1-q^j)$ is the Euler product function.

Type E_8

Theorem: If $C = (a_{ij})$ is the Cartan matrix of type E_8, then

$$\varphi(q^{30})^8 \prod_{j \equiv \pm 1, \pm 7, \pm 11, \pm 13 \pmod{30}} (1 - q^j)^{-1}$$

$$=$$

$$\sum_{\mathbf{n} \in \mathbb{Z}^8} q^{15 \mathbf{n} C \mathbf{n}^t - \sum_{i=1}^{8} n_i}$$

where $\varphi(q) = \prod_{j \geq 1} (1 - q^j)$ is the Euler product function.