Crossing Through and Bouncing Off ∞: Graphing Rational Functions

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Abstract

While rational functions are initially studied in high school, some of their behavior may be difficult to understand. For instance, when a rational function approaches a vertical asymptote, the graph may go to $+\infty$ and then either return from $+\infty$ or return from $-\infty$. But, where does it go and how does this happen? To explain this mathematical phenomenon, we consider topological techniques that wrap the real number line into a circle where $-\infty$ and $+\infty$ meet as one point. This then allows us to investigate principles of limits and local dominance in polynomial and rational functions. We also consider reciprocal relationships between zero and ∞ and polynomial and rational parent functions. Altogether, we connect all of these notions to better understanding vertical asymptotic behavior of rational functions.

1 Introduction

Consider the graph of the rational function below. At $x = -1$, $f(x)$ approaches $+\infty$ and then reappears to descend from $+\infty$. However, at $x = 1$, $f(x)$ approaches $-\infty$ and then reappears to descend from $+\infty$. Where do these graphs go and how does this happen?

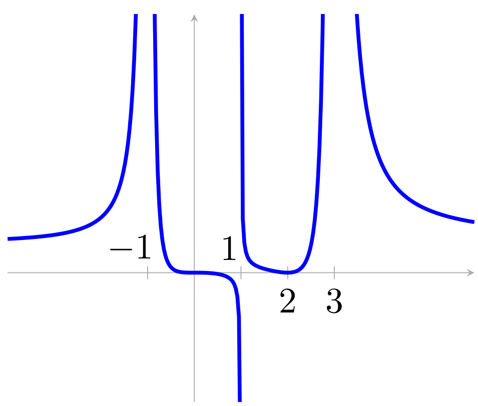

Figure 1: A rational function $f(x) = \dfrac{2x^3(x-2)^2}{(x+1)^2(x-1)(x-3)^2}$

Would you be surprised to find out that the behavior of the graph at the asymptote is connected to the behavior of polynomials at real roots? Would you like to use $\pm\infty$ as a value in a function, rather than simply limiting to it? What if the real number line was a circle, where $-\infty$ and $+\infty$ meet? We will consider these and other ideas as we investigate asymptotic behavior of rational functions in a manner that is readable by most high school students.

2 Big and Small

While ∞ is larger than any real number, it is not a real number. Therefore, arithmetic with ∞ can be problematic. Thus, for a moment, we will simply consider ∞ as "big" and 0 as "small". On our way to doing some arithmetic with ∞, let us recall some basic numerical facts. For example, the reciprocal of a very large (positive) number yields a very small (positive) number and vice-versa.

$$\frac{1}{\text{small}} = \text{big} \qquad \text{and} \qquad \frac{1}{\text{big}} = \text{small}$$

Figure 2: There is a duality between big and small.

Next, let a be a positive constant. Then the following facts are only some of those which hold: $\text{big} + a = \text{big}$; $\text{big} - a = \text{big}$; $\text{big} + \text{big} = \text{big}$; $a \cdot \text{big} = \text{big}$; $\text{big}/a = \text{big}$; $\text{big} \cdot \text{big} = \text{big}$; $\text{big}^a = \text{big}$; $a \cdot \text{small} = \text{small}$; $\text{big} - \text{small} = \text{big}$; $\text{big}/\text{small} = \text{big}$; and $\text{small}/\text{big} = \text{small}$.

In this vein, let us refer to a very small positive number (essentially zero from above) as 0^+ and a very small negative number (essentially zero from below) as 0^-. Likewise, we temporarily refer to an extremely large positive number as $+\infty$ and extremely large (in magnitude) negative number as $-\infty$. Then, since $1/\text{big} = \text{small}$ and $1/\text{small} = \text{big}$, we have

$$\frac{1}{0^+} = +\infty, \qquad \frac{1}{+\infty} = 0^+, \qquad \frac{1}{0^-} = -\infty, \qquad \text{and} \qquad \frac{1}{-\infty} = 0^-$$

Keeping in mind this relation to big and small numbers, we can see that statements such as: $\infty + 3 \cdot \infty = \infty$, $(-\infty)^2 + 0^- = \infty$, $(0^-)^4 + 5 \cdot 0^+ = 0^+$, and $-\infty/0^- = \infty$ are quite reasonable, whereas expressions such as: $\infty - \infty$, ∞/∞, $0^- + 0^+$, and $0^+/0^+$ must be ill defined. In more detail, $0^- + 0^+$ would be a very small negative number plus a very small positive number. This could be a very small positive or negative number, so we could say $0^- + 0^+ = 0^\pm$ but there must remain some ambiguity. Something like ∞/∞ is even more problematic. A very large number divided by another very large number could result in a large number, a small number, or anything in between: ∞/∞ might be ∞, 0^+, or any positive real number.

The real line is infinite in extent and the Cartesian coordinate plane extends to $\pm\infty$ in both the $x-$ and $y-$directions. We often wish to study polynomial functions whose domain is $(-\infty, +\infty)$ in the $x-$direction and whose graphs have far-left and far-right behavior approaching $\pm\infty$ in the $y-$direction. Beginning in high school, students investigate rational functions whose graphs may approach $\pm\infty$ near some vertical asymptote and may, seemingly mysteriously return from $\pm\infty$. With the notion of big and small established, we now wish to move to wrapping $\pm\infty$ in with the real numbers.

3 Compactification and Infinite Arithmetic

Infinity lies at the heart of much of modern mathematics. While it is difficult – dare we say impossible – to understand truly infinite objects and processes, there are various techniques which allow one to coax finiteness out of the infinite and thus make it more understandable. In particular, a technique called compactification allows us to complete asymptotes – yes, connect $-\infty$ to $+\infty$ – and make them look more like circles than lines. While both lines and circles might seem equally simple, circles are compact and lines are not. Without worrying too much about the technical definition of compactness, let us simply mention that compact sets behave much like finite sets and thus are easier to deal with. For a friendly introduction to the field of topology and the notion of compactness, consider [Cook, et al., 2016] and [Cook, et al., 2018].

Here we will consider two compactifications of the real line: $\mathbb{R}$. First, imagine the real line being bent into a circle. Then attach a point to connect the "ends" of the real line. Call that point $\pm\infty$ or merely ∞. Essentially this is the one point compactification of $\mathbb{R}$.

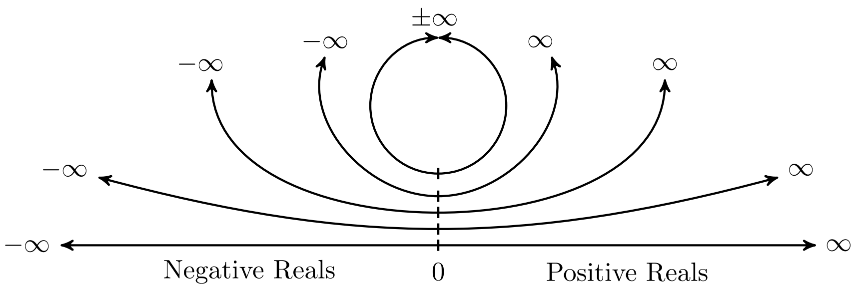

Figure 3: The one point compactification of the real numbers.

With this single point ∞ attached, $\mathbb{R} \cup \{\infty\}$ becomes topologically indistinguishable from a circle (topologically: up to bending and stretching). This one point compactification will be conceptually useful when graphing rational functions.

While we will see how this one point compactification assists our understanding of $\pm\infty$, we readily recognize that we do not typically use $\pm\infty$ in functions or calculations. For instance, rather than evaluating a function at ∞ (i.e., $f(\infty)$), we approach ∞ via a limit (i.e., $\lim\limits_{x\to\infty} f(x)$). In order to use $\pm\infty$ in our calculations and function notation, we propose another useful compactification of $\mathbb{R}$: the two point compactfication. This is obtained by attaching two extra points to $\mathbb{R}$ namely $+\infty$ and $-\infty$. So instead of $\mathbb{R} = (-\infty, \infty)$ we have $\mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. Now this extended real line will (topologically) behave exactly like a closed interval such as $[0, 1]$. Both circles and closed (bounded) intervals are compact and, thus, in many ways behave like finite sets.

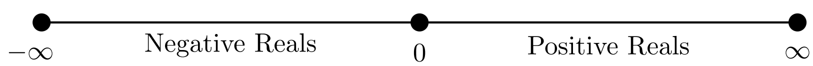

Figure 4: The two point compactification of the real numbers.

One advantage of using these compactifications is that we can evaluate functions and expressions at $\pm\infty$ rather than just limiting to these values. When we have a continuous function there is no difference between approaching the limit $\lim\limits_{x\to a} f(x)$ and evaluating $f(a)$. In other words, for a continuous function $f(x)$, $\lim\limits_{x\to a} f(x) = f(a)$. However, what about noncontinuous rational functions containing vertical asymptotes? As we will see, modifying the real line with these compactifications allows us to now investigate either $f(\infty)$ or when $f(x) = \infty$ and evaluate some expressions (e.g., $17 + \infty$, $2.7 - \infty$, $\infty + \infty$, $\infty \cdot \infty$), noting that $\infty - \infty$ and ∞/∞ remain ill defined.

Now imagine the xy-plane. We have infinities in all directions as either or both coordinates of (x, y) tend to $\pm\infty$. If we attach all of these infinities and gather them all together under the name ∞, our xy-plane becomes a sphere. When the xy-plane is used to represent the complex numbers, the corresponding sphere is called the Riemann sphere. Riemann and others used this one point compactification of the complex numbers to greatly advance the theory of complex functions.

Notice that on the Riemann sphere, $\pm\infty$ for both axes occurs at the north pole. Thus, as we will see, as a function approaches $\pm\infty$ the graph can either pass through $\pm\infty$ at the north pole (e.g., pass from the negative side of ∞ to the positive side of ∞ or vice versa) or bounce off $\pm\infty$ (e.g., approach ∞ from the positive side, bounce off ∞, and then return on the positive side of ∞). Understanding this behavior will help us to read and interpret some unique graphs which follow.

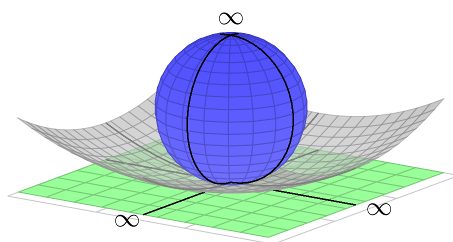

Figure 5: Compactifying the plane.

4 Dominance and Infinite Arithmetic

Two concepts regarding infinity are connected: at times we are approaching $\pm\infty$ and at other times we getting infinitesimally close to a value, and each of these approachments can occur at different speeds. For instance, for $a > 1$ and $b > 0$, as a and b grow, a^b grows to ∞. However, a^b grows faster by changing b than by changing a. Similarly, for $0 < a < b < 1$ and $1 < c$, as c grows, $f(c) = b^c$ approaches 0, but $g(c) = a^c$ approaches 0 more quickly.

For the real polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_k x^k$, the dominant term (which most affects the value of the function) is dependent upon whether we are approaching ∞ or 0. When $x \to \infty$, $a_n x^n$ approaches $\pm\infty$ much more quickly than the other terms and thus dominates. When $x \to 0$, all non-constant terms become 0. If $k = 0$, the polynomial approaches $a_0 x^0 = a_0$, so this constant term dominates. When $k > 0$, the polynomial does tend to 0, but notice that the bottom term $a_k x^k$ goes to zero slower than the higher order terms. Thus as x goes to 0, the bottom term $a_k x^k$ always dominates. Restating this in the context of limits, we can provide an example: $\lim\limits_{x\to\infty} \frac{1}{x^3-5x^2+1} = \lim\limits_{x\to\infty} \frac{1}{x^3} = \frac{1}{+\infty} = 0$ since the x^3 term dominates. In fact, we have that our limit is 0^+ in the sense that we are approaching zero from the positive side. So if $f(x) = \dfrac{1}{x^3 - 5x^2 + 1}$, we can write $f(\infty) = 0^+$ or just $f(\infty) = 0$.

As a few more examples:

$$\lim_{x\to\infty} -6x^4 + 99x^3 - 7x^2 - 4 = \lim_{x\to\infty} -6x^4 = -\infty,$$

$$\lim_{x\to\infty} \frac{3x^2 - 5x + 2}{-6x^3 + 17x^2 + 3x - 1} = \lim_{x\to\infty} \frac{3x^2}{-6x^3} = \lim_{x\to\infty} \frac{1}{-2x} = \frac{1}{-\infty} = 0 \text{ or specifically } = 0^-, \qquad \text{and}$$

$$\lim_{x\to\infty} \frac{10x^4 - 3x^2 - 112}{5x^4 - 13x^3 + 8x} = \lim_{x\to\infty} \frac{10x^4}{5x^4} = \lim_{x\to\infty} 2 = 2.$$

Likewise,

$$\lim_{x\to 0} 2x^5 + 67x^2 - 5x = \lim_{x\to 0} -5x = 0 \qquad \text{and}$$

$$\lim_{x\to 0} \frac{7x^2 - 10x}{9x^4 - 8x^3 + 2x} = \lim_{x\to 0} \frac{-10x}{2x} = \lim_{x\to 0} \frac{-10}{2} = -5.$$

If we are careful to specify how we approach 0 (i.e., from the positive or negative side), we can evaluate examples like:

$$\lim_{x\to 0^+} \frac{13x^4 - 5x}{x^3 - 2x^3 + 3x^2} = \lim_{x\to 0^+} \frac{-5x}{3x^2} = \lim_{x\to 0^+} \frac{-5}{3x} = \frac{-5}{0^+} = -\infty.$$

Notice that as $x \to \pm\infty$, if the numerator has the term of greatest degree, the function approaches $\pm\infty$, and if the denominator has the term of largest degree, we head to 0. If there is a tie, we head toward the ratio of the leading coefficients. As $x \to 0$, we have a dualized behavior which is explained below.

In this section, in order to introduce the notion of dominance, we employed limit notation. If we revert to using the compactifications defined above, we can use the values of 0 and $\pm\infty$. However, we need a word of caution here. While after compactifying we can use $\pm\infty$ as inputs and values, we may not be able to use our original formula for our rational function. For example: $\frac{3\cdot\infty^2 - 5\cdot\infty + 2}{-6\cdot\infty^3 + 17\cdot\infty^2 + 3\cdot\infty - 1}$ is ill defined. However, if we first remove non-dominant terms and simplify, we can get a formula which allows evaluation. For example, when evaluating at $x = \infty$, $f(x) = \frac{3x^2 - 5x + 2}{-6x^3 + 17x^2 + 3x - 1}$ is dominated by $\frac{3x^2}{-6x^3} = -\frac{1}{2x}$. So $f(\infty) = -\frac{1}{2\cdot\infty} = -\frac{1}{\infty} = 0^-$. Likewise, $f(x) = \frac{10x^4 - 3x^2 - 112}{5x^4 - 13x^3 + 8x}$ becomes $\frac{10x^4}{5x^4} = \frac{10}{5} = 2$ near $x = \infty$, so $f(\infty) = 2$. So after compactifying we can calculate with $\pm\infty$, but only with accommodating formulas.

Compactification also allows us to summarize the results of using ∞ and 0 in rational functions: For $f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_k x^k}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_\ell x^\ell}$, assume a_n, a_k, b_m, b_ℓ are all nonzero. If $n > m$, then $f(\infty) = \pm\infty$; if $n < m$, then $f(\infty) = 0$; and if $n = m$, then $f(\infty) = \frac{a_n}{b_m}$. If $k > \ell$, then $f(0) = 0$; if $k < \ell$, then $f(0) = \pm\infty$; and if $k = \ell$, then $f(0) = \frac{a_k}{b_\ell}$.

5 Local Dominance, Crossing or Bouncing off $\pm\infty$, and Rational Functions

In this section, we will connect a number of ideas. First, we consider the behavior of a polynomial function to investigate the nature of local dominance. Second, we connect the notion of the graph of a function either crossing or bouncing off the $x-$axis with either crossing or bouncing off ∞. We will then connect these ideas by investigating the behavior of a rational function.

We begin by investigating the local behavior about real roots of polynomials. Consider $f(x) = (x-1)^3(x-5)^4$. See the Figure 6 below. As x approaches 1, $(x-5)^4$ becomes $(1-5)^4 = 256$. Thus, when $x \approx 1$, we have $f(x) \approx 256(x-1)^3$. While 256 may seem relatively large, 256 is merely a constant. However, when $x \approx 1$, $(x-1) \to 0$, or $(x-1) \to$ small. Previously we agreed that constant $\cdot$ small^3 = small. In other words, near $x = 1$ the factor $(x-1)^3$ locally dominates and the entire product becomes small = 0. Similarly, at $x \approx 5$, $f(x) \approx (5-1)^3(x-5)^4 = 64 \cdot (\text{small})^4 = \text{small} = 0$, as the factor $(x-5)$ locally dominates. As we move to the left away from $x = 1$ and to the right away from $x = 5$, our polynomial $f(x) = (x-1)^3(x-5)^4 = x^7 - 23x^6 + 213x^5 - 1011x^4 + 2595x^3 - 3525x^2 + 2375x - 625$ is dominated by x^7. Thus $f(\infty) = (\infty)^7 = \infty$ and $f(-\infty) = (-\infty)^7 = -\infty$. Note that we can avoid distributing the factored polynomial by asserting dominance within each factor: For large x values, $f(x) = (x-1)^3(x-5)^4 \approx (x)^3(x)^4 = x^7$.

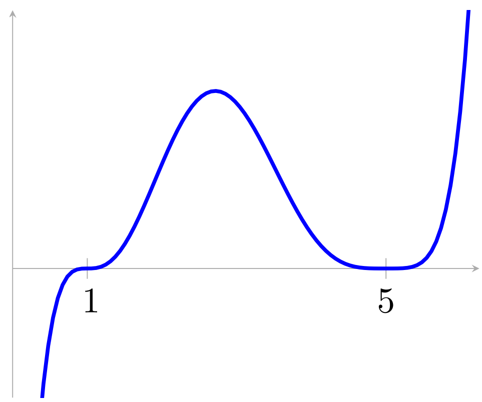

Figure 6: A polynomial function $f(x) = (x-1)^3(x-5)^4$

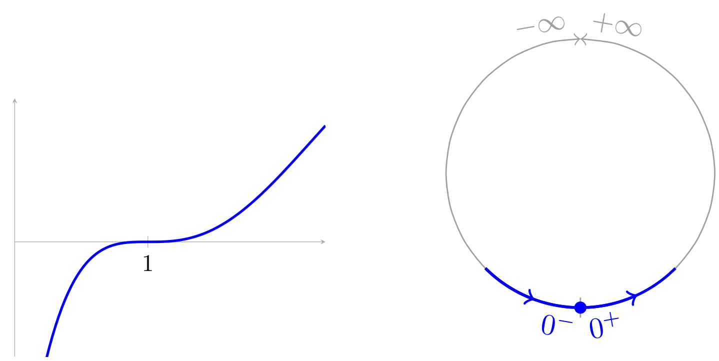

Figure 6A: Root behavior of $f(x) = (x-1)^3(x-5)^4$ at $x = 1$

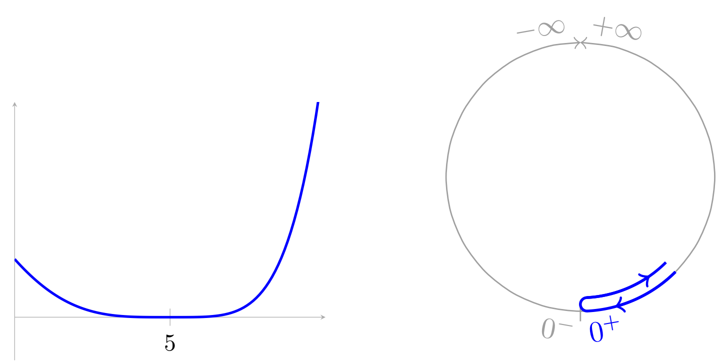

Figure 6B: Root behavior of $f(x) = (x-1)^3(x-5)^4$ at $x = 5$

Now that we know that $f(x) = (x-1)^3(x-5)^4$ has far-left behavior approaching $-\infty$, real zeros at $x = 1$ and $x = 5$, and far-right behavior approaching ∞, we can investigate how it approaches these values. In fact: as $x \to 1$, $f(x)$ behaves like $y = x^3$ near the origin; as

$x \to 5$, $f(x)$ behaves like $y = x^4$ near the origin; and as $x \to \pm\infty$, $f(x)$ behaves similarly to $y = x^7$. Understanding the behavior of a function as it approaches a real root or $\pm\infty$ allows us to decompose our understanding of the function to polynomial parent functions: $y = x^n$.

Capitalizing on the use of polynomial parent functions also leads to a number of connected ideas. First, we know that $y = x^{\text{odd}}$ passes through the $x-$axis and that $y = x^{\text{even}}$ bounces off the $x-$axis. Second, polynomial parent functions can be readily compared to their rational parent function counterparts. For instance: $y = x \longleftrightarrow y = \frac{1}{x}$; $y = x^2 \longleftrightarrow y = \frac{1}{x^2}$; $y = x^3 \longleftrightarrow y = \frac{1}{x^3}$; and $y = x^4 \longleftrightarrow y = \frac{1}{x^4}$. However, connecting these two ideas gives rise to a wonderful mathematical observation: As $y = x^{\text{odd}}$ passes through the $x-$axis, $y = \frac{1}{x^{\text{odd}}}$ passes through $\pm\infty$, and as $y = x^{\text{even}}$ bounces off the $x-$axis, $y = \frac{1}{x^{\text{even}}}$ bounces off $\pm\infty$. Let's see what this looks like.

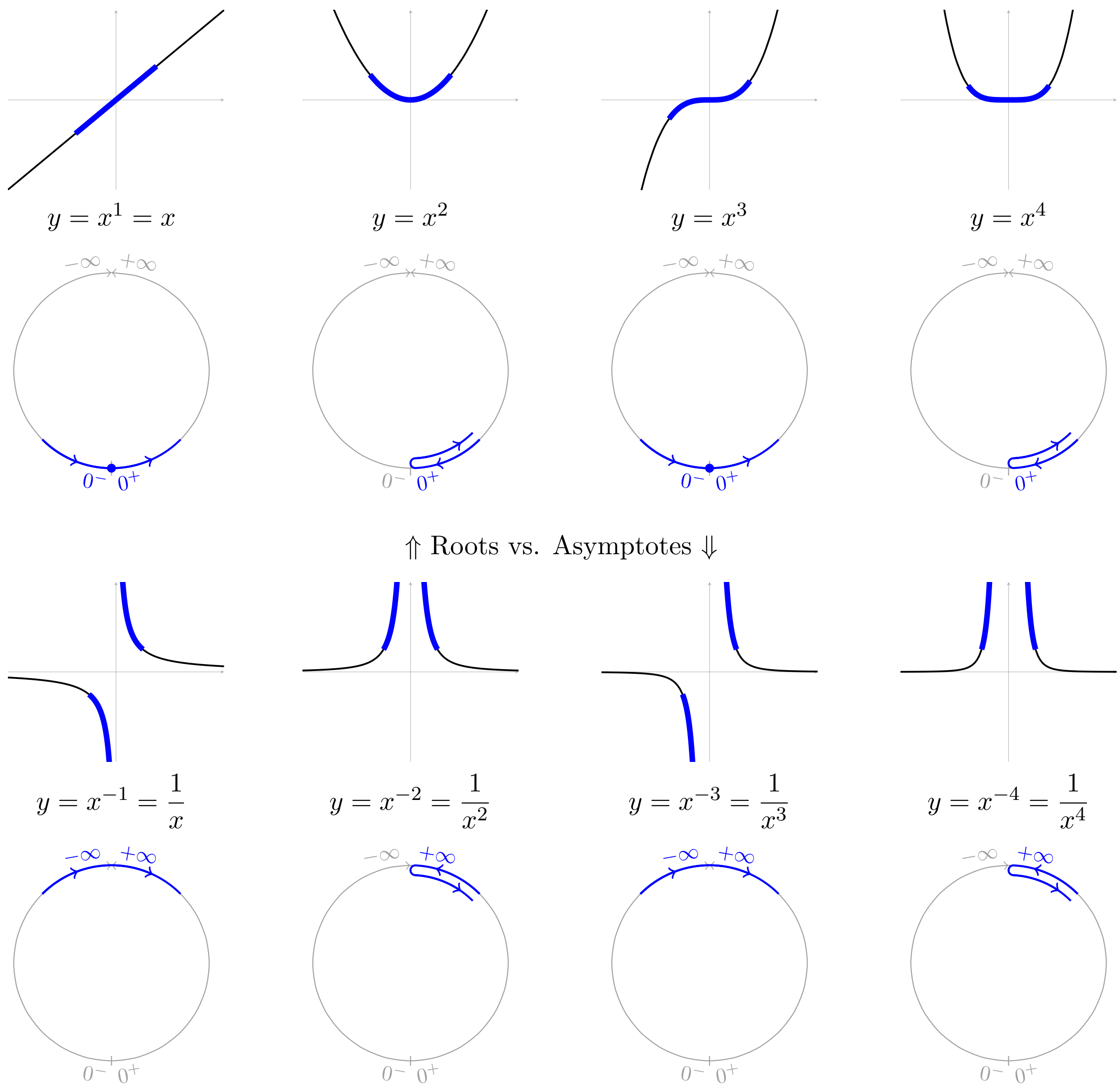

Figure 7: Root and asymptote behavior

As a culminating example, we tie together all aspects previously mentioned through the context of a rational function. Let's consider the rational function $f(x) = \dfrac{-2(x-1)}{(x+1)(x-2)^2}$. We have a root at $x = 1$ corresponding to the factor $(x-1)$ (i.e., the factor $(x-1)$ has local dominance near $x = 1$), thus we cross the x-axis at $x = 1$ similar to $y = x$ at $x = 0$. Next, we have poles at $x = -1$ and $x = 2$. When $x \approx -1$ (and the factor $(x+1)$ in the denominator has local dominance), we have $f(x) \approx \dfrac{-2(-1-1)}{(x+1)(-1-2)^2} = \dfrac{4}{9(x+1)}$. Thus, $f(x)$ has a vertical asymptote $x = -1$ just like $y = 1/x$ does at $x = 0$, and we pass through $\pm\infty$. When $x \approx 2$ (ascribing local dominance to the $(x-2)$ factor in the denominator), we have $f(x) \approx \dfrac{-2(2-1)}{(2+1)(x-2)^2} = \dfrac{-2}{3(x-2)^2}$. Thus, $f(x)$ has a vertical asymptote $x = 2$ just like $y = -1/x^2$ does at $x = 0$, and we bounce off of $-\infty$. Finally, as $x \to \pm\infty$, we have $f(x) \approx \dfrac{-2(x)}{(x)(x)^2} = \dfrac{-2}{x^2}$, with the ratio $\dfrac{-2}{x^2}$ possessing dominance. Thus $f(\pm\infty) = -2/(\pm\infty)^2 = -2/\infty = 0$ (or more precisely 0^-). Thus $f(x)$ has a horizontal asymptote $y = 0$. More than that, $f(x)$ approaches this asymptote from the negative side (both as x heads to ∞ and $-\infty$).

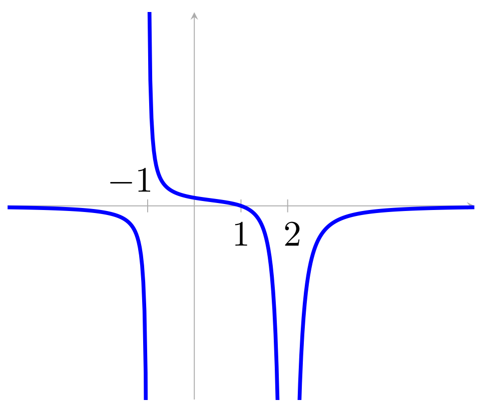

Figure 8: A rational function $f(x) = \dfrac{-2(x-1)}{(x+1)(x-2)^2}$

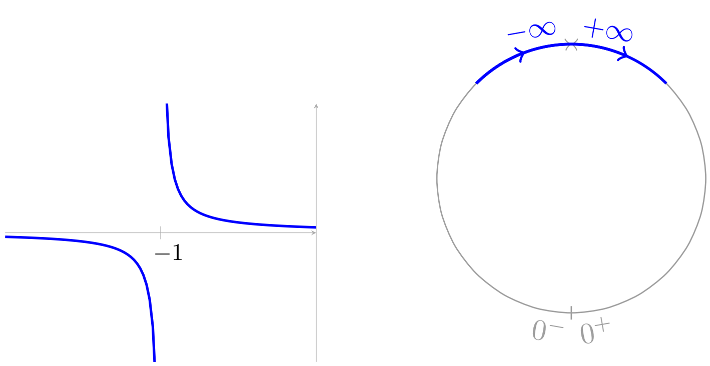

Figure 8A: Asymptote behavior of $f(x) = \dfrac{-2(x-1)}{(x+1)(x-2)^2}$ at $x = -1$

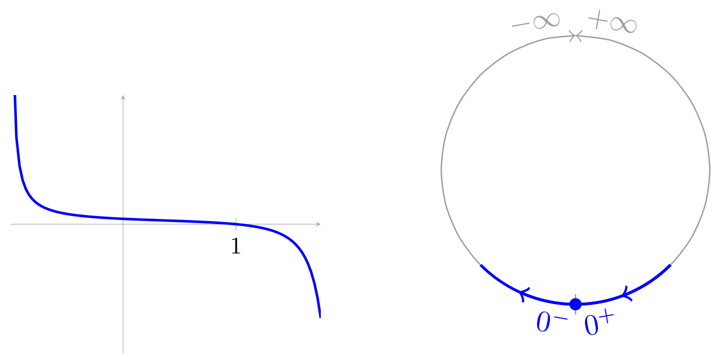

Figure 8B: Root behavior of $f(x) = \dfrac{-2(x-1)}{(x+1)(x-2)^2}$ at $x = 1$

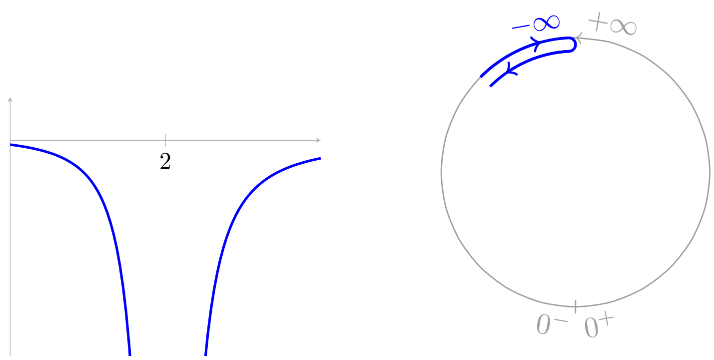

Figure 8C: Asymptote behavior of $f(x) = \dfrac{-2(x-1)}{(x+1)(x-2)^2}$ at $x = 2$

6 Individual Investigations

Here we provide some suggested individual investigations.

1. Investigate the function $f(x) = -5x^2(x+1)(x-2)^4$ in respect to crossing through or bouncing off 0. Compute and interpret: $f(\pm\infty)$.

2. Investigate the function $f(x) = \dfrac{2x^3(x-2)^2}{(x+1)^2(x-1)(x-3)^2}$ in respect to crossing through or bouncing off 0 and $\pm\infty$. Compute and interpret: $f(\pm\infty)$, $f(-1)$, $f(1)$, and $f(3)$.

3. Write a single rational function in factored form such that: (a) it passes through the $x-$axis at $x = -1$; (b) it bounces off the $x-$axis at $x = 2$; (c) it bounces off ∞ at $x = 1$; (d) it passes through $\pm\infty$ at $x = 3$; and (e) has a horizontal asymptote at $y = 2$.

4. Investigate other functions as in problems 1 and 2.

5. Create your own conditions as in problem 3 and write appropriate functions.

7 Conclusion

Looking back, we have covered much mathematics: big and small numbers; compactification, making $\pm\infty$ more concrete and usable; limits and local dominance; connecting polynomial parent functions with their counterpart parent rational function; crossing or bouncing off $\pm\infty$, and rational functions. We did so by topologically wrapping the $x-$ and $y-$axes into circles sharing the point $(0, 0)$. In essence, and which was our purpose, we used advanced mathematical techniques in ways which bring light to high school mathematics. We hope that you enjoyed reading this paper as much as we enjoyed writing it. Hopefully, this valuable and interesting bit of mathematics ignites your curiosity. For a somewhat informal introduction to basic topological concepts, consider [Cook, et al., 2016] and [Cook, et al., 2018]. For a careful, technical treatment, we recommend [Munkres, 2000].

References

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