Types of Infinity

William J. Cook & Michael J. Bossé

July 20, 2019

Abstract

The concept of infinity pervades much of modern mathematics. Even elementary school students use the word infinity. However, what do we mean by infinity? While, for many infinity merely means either very large or not finite, in various contexts, the term infinity carries more information. Here we consider infinity in the contexts of cardinality (sizes of sets), ordinals (the order of numbers), and real numbers (a number system). Our aim is to introduce these various notions of infinity to interested mathematics students and whet their appetite for more.

1 Introduction

Teachers and students of mathematics can appreciate Buzz Lightyear's mathematical prowess and insight as he correctly states, "To infinity and beyond." Indeed, as we will see, Buzz gets it!

Even in elementary school, students use the word *infinity* to denote the notion of very large. Later in mathematics, students discuss infinite sets as well as infinities related to number systems. But, do we all mean the same thing by *infinity*? As we will see, infinity has a number of distinct yet interconnected meanings. For instance, infinity can be explored in the contexts of cardinals (sizes of sets), ordinals (orderings of sets), and real numbers (a number system). In this paper, we we present these ideas in a manner commensurate with the interests of inquisitive high school and undergraduate students and teachers. We hope that these ideas regarding infinity stimulate the reader to new interests in these intriguing concepts.

This paper is a soft investigation of concepts regarding infinity. For more detailed exposition, consider the following: [Halmos, 2011] and [Roitman, 1990]. In our companion paper [Cook et al., 2019] we consider the concept of ∞ and employ one- and two-point compactifications to investigate how the graphs of rational functions can pass through or bounce off ∞ in a manner similar to the graph of a polynomial passing through or bouncing off the x-axis.

2 Cardinal Numbers

Most simply stated, the cardinality of a set is the number of elements in that set. Let us begin with a definition which will be followed up with a couple of examples.

Definition 2.1. *Let A and B be sets. We say that A and B have the same* cardinality*, denoted* $\mathrm{card}(A) = \mathrm{card}(B)$*, if and only if there is a one-to-one and onto function (i.e., invertible function) whose domain is A and range is B.*

A function $f : A \to B$ is one-to-one if each element of A maps to a different element of B, though there may be some extra elements of B which are not partnered with elements of A. A function $f : A \to B$ is onto if each element of B is mapped to by an element of A, though there may be some elements of B which are partnered with several elements of A. A function is invertible if it is both one-to-one and onto. Such a function makes sure every element of A is uniquely partnered with some element of B and vice-versa.

This means that sets of objects have the same size (i.e., cardinality) if we can match the elements between the sets in a one-to-one fashion. For example, $P = \{a, b, c\}$ and $Q = \{☺, ☹, \otimes\}$ both have the same cardinality: $\text{card}(P) = \text{card}(Q)$ because we can exhibit an invertible function between these sets: $a \leftrightarrow ☺$, $b \leftrightarrow ☹$, $c \leftrightarrow \otimes$. More simply stated: every element in set P maps to a distinct element in set Q and every element in Q maps to a distinct element in P (resulting in no unused elements in either direction). In this case, we might write $\text{card}(P) = \text{card}(Q) = 3$.

When cardinalities are not the same, we can still compare them with inequalities. We write that $\text{card}(A) \leq \text{card}(B)$ if there is function $f : A \to B$ which is one-to-one but possibly not onto. If there is a one-to-one function $f : A \to B$ but there is no invertible function between A and B, we write $\text{card}(A) < \text{card}(B)$. We might want to restate this as: If set B has at least one more element than set B, then $\text{card}(A) < \text{card}(B)$. However, while this is true for finite sets, it may fail for infinite sets. When a set is infinite, adding in one more element does not change its cardinality!

It is easy to show[1] that if one has an onto function $f : A \to B$, then there must be a one-to-one function $g : B \to A$. This means that one-to-one and onto are kind of dual to each other. An intuitive, yet a bit tricky to prove, theorem says that if $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(A)$, then $\text{card}(A) = \text{card}(B)$ (i.e., if neither set is bigger than the other, they must have the same size). This result is known as the Cantor-Schröder-Bernstein theorem. It was published by Cantor without proof and then later proven by Schröder and then reproven independently by Bernstein.

To continue investigating the notion of infinite cardinalities, we must first define the relation of proper subset. Set A is a proper subset of set B, denoted $A \subsetneq B$ if every every element of A is an element of B and there is at least one element in B which is not in A. For finite sets, this would automatically imply that $\text{card}(A) < \text{card}(B)$. However, let us investigate this a little more. Consider the sets $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{E} = \{0, 2, 4, \dots\}$. While it may be tempting to say, "$\mathbb{N}$ has twice as many elements as $\mathbb{E}$, since $\mathbb{E}$ has none of the odd numbers in $\mathbb{N}$", this may not be correct. Consider the the function $f : \mathbb{N} \to \mathbb{E}$ defined by $f(n) = 2n$. Since f is invertible, $\text{card}(\mathbb{N}) = \text{card}(\mathbb{E})$. However, $\mathbb{E} \subsetneq \mathbb{N}$. While this may seem like a contradiction, it leads directly to the *definition* of infinite sets.

Definition 2.2. *Suppose $A \subsetneq B$ and there is a function $f : A \to B$ which is both one-to-one and onto. Then B is an* infinite *set. Indeed, both A and B are infinite sets. In other words, a set is infinite when it has the same cardnality as one of its proper subsets.*

Let us consider another example to help solidify this notion. We create a mapping from $\mathbb{N} = \{0, 1, 2, \dots\}$ to $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Immediately, we notice that $\mathbb{N} \subsetneq \mathbb{Z}$. Again, it may seem that $\mathbb{Z}$ has more than twice as many elements as $\mathbb{N}$. But, again, this is not correct. Consider the the function $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n+1)/2 & \text{if } n \text{ is odd} \end{cases}$ between $\mathbb{N}$ and $\mathbb{Z}$. It is

[1] For each $x \in \text{range}(f) = B$ choose some $g_x \in A$ such that $f(g_x) = x$ (can be done since f is onto). Then $g : B \to A$ given by $g(x) = g_x$ is a one-to-one function (if $g_x = g(x) = g(y) = g_y$ then $x = f(g_x) = f(g_y) = y$).

not hard to show that f is invertible, so $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})$. Therefore, by the definition, both $\mathbb{N}$ and $\mathbb{Z}$ are infinite sets.

Before we investigate another example, we propose another definition which will be helpful.

Definition 2.3. *We say that a set X is* countable *if* $\text{card}(X) \leq \text{card}(\mathbb{N})$. *When* $\text{card}(X) = \text{card}(\mathbb{N})$ *we say X is* countably infinite. *When* $\text{card}(X) < \text{card}(\mathbb{N})$ *we have that X is a finite set. If* $\text{card}(X) > \text{card}(\mathbb{N})$, *then X is* uncountable.

Notice that countable sets are exactly those which can be (in theory) listed off. If $\text{card}(X) = \text{card}(\mathbb{N})$, then there is an invertible function $f : \mathbb{N} \to X$. So for each $x \in X$ there is a unique $n \in \mathbb{N}$ such that $f(n) = x$ (n exists because f is onto, it is unique because f is one-to-one). Thus we can write $x = f(n) = x_n$. In other words, $X = \{x_0, x_1, x_2, \dots\}$. Conversely, if we can list off a set $X = \{x_0, x_1, x_2, \dots\}$, then we get a one-to-one function $f : X \to \mathbb{N}$ defined by $f(x_n) = n$, so that $\text{card}(X) \leq \text{card}(\mathbb{N})$ (with equality if X is infinite).

Now let's consider another mapping which may seem even more impossible. Recall that the rational numbers are defined as $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$. Is it possible that there is a one-to-one and onto function mapping between $\mathbb{N}$ and $\mathbb{Q}$? Since $\mathbb{N}$ is sparse on the number line (i.e., for any $n \in \mathbb{N}$, there are no natural numbers between n and $n+1$) and $\mathbb{Q}$ is dense (i.e., for any distinct $a, b \in \mathbb{Q}$, there exists an infinite number of rational numbers between a and b), it certainly seems improbable that such a mapping may exist. However, let us look at the figure below.

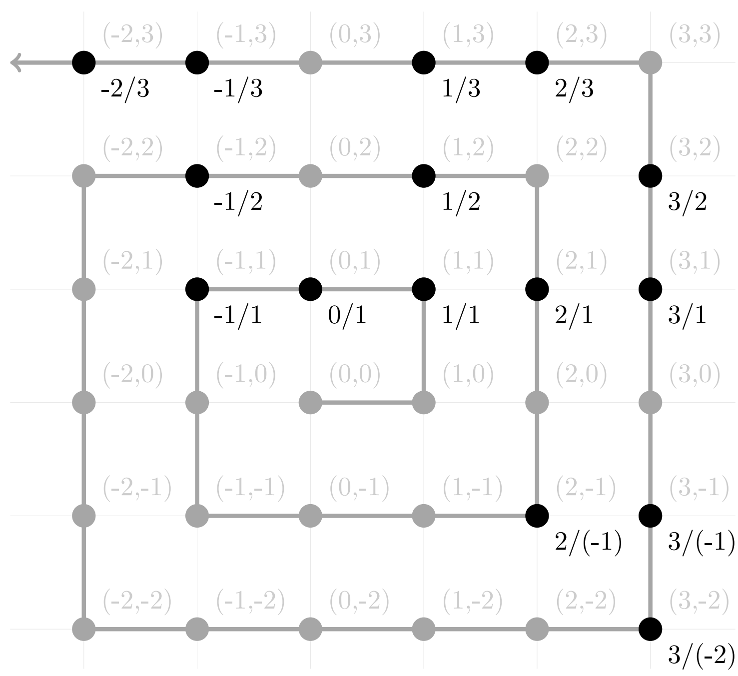

Figure 1: Listing off the rational numbers.

Here we have a plot of all points in the plane with integer coordinates. We start at the origin and spiral outward as shown above. Notice that the spiral hits a sequence of coordinates which can be likened to fractions: $(x, y) \to \frac{x}{y}$. To get the enumeration we desire, we need to make two additional rules. First, we skip over the x-axis since $x/0$ is not a rational number. Second, if we have already encountered a corresponding rational number, we skip over that redundant point.

Following these rules, we obtain a list $1, 0, -1, -2, 2, 1/2, -1/2$ etc., so the mapping $0 \mapsto 1$, $1 \mapsto 0$, $2 \mapsto -1$, etc. gives us a function from $\mathbb{N}$ to $\mathbb{Q}$. This function is one-to-one because we skip over any rational number we have already seen, and it is onto because every rational number can be expressed as x/y for some integers x and y (i.e., we will eventually see any element of $\mathbb{Q}$). Therefore, $\mathbb{N}$ and $\mathbb{Q}$ are both infinite and $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Q})$. This means that $\mathbb{Q}$ is countable – although we suggest that you not try doing so. However, if you did, it is no more difficult than counting all the elements in $\mathbb{N}$. We might also note that this way of listing $\mathbb{Q}$, unlike the way we list $\mathbb{N}$, has nothing to do with the natural $<$ ordering of real numbers.

It is now time to name $\text{card}(\mathbb{N})$. We call it it Aleph null or $\aleph_0$ (Aleph is the first letter of the Hebrew alphabet). In fact, $\aleph_0$ is the smallest cardinality for infinite sets. If we allow $\mathbb{A}$ to denote that algebraic numbers (i.e., the set of all roots of polynomials with rational coefficients), we can state that $\text{card}(\mathbb{N}) = \text{card}(\mathbb{E}) = \text{card}(\mathbb{Z}) = \text{card}(\mathbb{Q}) = \text{card}(\mathbb{A}) = \aleph_0$. In other words, $\mathbb{N}$, $\mathbb{E}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{A}$ (as well as many other sets) are all infinite with the same cardinality of $\aleph_0$.

It would now be quite intuitive to believe that there is only one cardinality for all infinite sets. In order to investigate this further, let us define power sets. Given any set X, its power set, $\mathcal{P}(X)$ is the set of all of the subsets of X. For instance, let $X = \{a, b, c\}$. Then $\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ where $\emptyset = \{\}$ is the empty set (i.e., the set with no elements). A few more examples would quickly verify that, if $\text{card}(X) = n$, then $\text{card}(\mathcal{P}(X)) = 2^n$. This leads immediately to Cantor's famous theorem.

Theorem 2.4. *Let X be any set and let $\mathcal{P}(X)$ denote the power set of X. Then* $\text{card}(X) < \text{card}(\mathcal{P}(X))$. *In other words, the power set of X is always larger than X itself.*[2]

Cantor's theorem tells us that for any set X: $\text{card}(X) < \text{card}(\mathcal{P}(X)) < \text{card}(\mathcal{P}(\mathcal{P}(X))) < \cdots$ and so, in particular, $\text{card}(\mathbb{N}) < \text{card}(\mathcal{P}(\mathbb{N})) < \text{card}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) < \cdots$. Since $\text{card}(\mathbb{N}) = \aleph_0$ is countable infinity, $\text{card}(\mathcal{P}(\mathbb{N}))$ must an uncountable infinity. This also establishes that there are infinitely many distinct infinite cardinalities and that no matter how big our set is, its power set is even larger. What is truly intriguing – and beyond the scope of this paper – is that some collections are too large to be sets! There is no such thing as the "set of all sets". Assuming the existence of such an object leads to a contradiction. We leave it to the interested reader to investigate this further [Halmos, 2011]. Wait! Did you see it? If $\aleph_0 = \text{card}(\mathbb{N}) < \text{card}(P(\mathbb{N}))$, then there must be something larger than $\aleph_0$. Let us now use Cantor's diagonalization argument to show that $\mathbb{R}$ is uncountable (i.e., $\text{card}(\mathbb{N}) < \text{card}(\mathbb{R})$).

Theorem 2.5. *The set of real numbers is uncountable.*

Proof: For sake of contradiction, suppose that $\mathbb{R}$ is countable. Therefore, every subset of $\mathbb{R}$ is countable and in particular, $I = [0, 1)$ is countable. This means we can list the elements of I say $I = \{x_1, x_2, x_3, \dots\}$.

Next, every real number has a decimal expansion. In fact, it has a unique decimal expansion if we do not allow trailing 9's (for example, $12.3\overline{9} = 12.4$). Consider $x_j \in I$ so that $0 \le x_j < 1$. Expand x_j (without trailing 9's) and get $x_j = 0.d_{j1}d_{j2}d_{j3}\cdots = d_{j1}10^{-1} + d_{j2}10^{-2} + d_{j3}10^{-3} + \cdots$ where each digit $d_{ij} \in \{0, 1, \dots, 9\}$. Now focus on the i^{th} decimal digit of the i^{th} number in our

[2] The proof of this theorem (which we omit) is surprisingly simple and uses an ingenious "self-referencing" trick.

list:

$$\begin{array}{rcl} x_1 & = & 0.d_{11} d_{12}\ d_{13}\ d_{14}\ldots \\ x_2 & = & 0.\,d_{21} d_{22} d_{23}\ d_{24}\ldots \\ x_3 & = & 0.\,d_{31}\ d_{32} d_{33} d_{34}\ldots \\ & \vdots & \end{array}$$

Define y_i to be 1 if $y_i = 0$ and $y_i = d_{ii} - 1$ otherwise. Then $y = 0.y_1y_2y_3\ldots$ is a real number in $I = [0, 1)$. We chose y's digits so that it does not end in trailing 9's (this also keeps us from getting $0.\overline{9} = 1$).

Notice that $y \neq x_i$ for each $i = 1, 2, \ldots$ since y and x_i have differing i^{th} digits. Thus y is not on the list and so our list is incomplete (contradiction). ♦

Another way to establish that $\mathbb{R}$ is uncountable is to show $\text{card}(\mathbb{R}) = \text{card}(\mathcal{P}(\mathbb{N}))$.[3] This again shows that $\mathbb{R}$ is strictly larger than $\mathbb{N}$. We call the cardinality of the real numbers *continuum* and denote it by $\text{card}(\mathbb{R}) = \text{card}(\mathcal{P}(\mathbb{N})) = 2^{\aleph_0}$.

Again, recall that $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of integers and since we can list them off: $0, -1, 1, -2, 2, -3, 3, \ldots$ we know that $\mathbb{Z}$ is countable. As we have seen, $\mathbb{Q}$ is also countable, as we already listed them off. In fact, from our enumeration of $\mathbb{Q}$ we may have already anticipated that $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(p, q) \mid p, q \in \mathbb{Z}\}$ is countable: $\text{card}(\mathbb{Z}^2) = \text{card}(\mathbb{N})$. To show this concretely, plot the elements of $\mathbb{Z}^2$ as grid points in the plane and then list them off by spiraling outward.

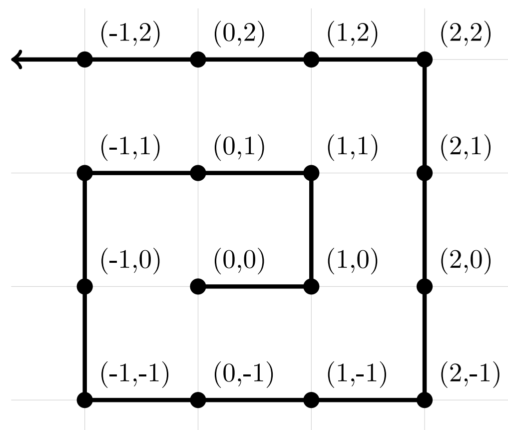

Figure 1: $\mathbb{Z}^2 = \{(0, 0), (1, 0), (1, 1), (0, 1), (-1, 1), \ldots\}$ is countable.

Similarly, recalling that $\mathbb{N} = \{0, 1, 2, \ldots\}$, one can show that $f : \mathbb{N}^2 \to \mathbb{N}$ defined by $f(x, y) = \dfrac{1}{2}(x^2 + 2xy + y^2 + 3x + y)$ is an invertible function so that $\text{card}(\mathbb{N}^2) = \text{card}(\mathbb{N})$.

More generally, for non-empty sets X and Y where at least one is infinite, one can show that the cardinality of $X \times Y$ is the same as the maximum of the cardinalities of X and Y (this is

[3] Without getting into the details of such a proof, here is the idea: First, one can find an invertible function between $\mathbb{R}$ and the interval $I = [0, 2]$ so that $\mathbb{R}$ and I have the same cardinality. Next, each element of $b \in I$ can be represented in a binary expansion: $b = b_0.b_1b_2\cdots = b_0 2^0 + b_1 2^{-1} + b_2 2^{-2} + \cdots$ each binary digit b_i being either 0 or 1. Create a set $B = \{k \in \mathbb{N} \mid b_k = 1\}$. So each element of I is associated with a subset of $\mathbb{N}$. With a little effort one can show this association is one-to-one and onto.

a significantly more difficult result, see [Roitman, 1990]). For example, $\text{card}(\mathbb{R} \times \mathbb{Z}) = \text{card}(\mathbb{R})$ and $\text{card}(\mathbb{R}^2 \times \mathbb{Q}) = \text{card}(\mathbb{R}^2) = \text{card}(\mathbb{R})$. It is also true that the cardinality of the union of two sets is the same as the maximum of their cardinalities as long as at least one of them is infinite. Consider the irrational numbers, $\mathbb{I} = \mathbb{R} - \mathbb{Q} = \{r \in \mathbb{R} \mid r \notin \mathbb{Q}\}$. Remembering that $\mathbb{R}$ is uncountably infinite and $\mathbb{Q}$ is countably infinite, $\mathbb{R}$ must get its larger size from $\mathbb{I}$. In more detail, there are infinitely many irrationals, so $\text{card}(\mathbb{I}) \geq \text{card}(\mathbb{Q})$ since the rationals are countably infinite and that is the smallest kind of infinite cardinal. This means that $\text{card}(\mathbb{R}) = \text{card}(\mathbb{I} \cup \mathbb{Q}) = \max\{\text{card}(\mathbb{Q}), \text{card}(\mathbb{I})\} = \text{card}(\mathbb{I})$. Thus, there are uncountably many irrational numbers. Indeed, $\text{card}(\mathbb{I}) = \text{card}(\mathbb{R}) = \text{card}(\mathbb{C})$, where $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ is the set of complex numbers.

One can also show that a union of countably many countable sets is itself countable: $A = A_1 \cup A_2 \cup A_3 \cup \cdots$ is countable if each A_n is countable. Likewise, the collection of all *finite* subsets of a countable set is still countable. So while $\mathcal{P}(\mathbb{N})$ is uncountable (in fact, continuum in size), the set of finite subsets $\mathcal{P}' = \{X \subsetneq \mathbb{N} \mid X \text{ is finite }\}$ is countable.

Let us temporarily leave the notion of cardinality and consider infinity through the context of ordinal numbers.

3 Ordinal Numbers

While cardinals measure size, ordinals keep track of orderings – specifically well orderings. We begin with some background regarding ordering and then we will construct the collection of ordinal numbers.

With respect to orderings, infinite sets behave in a very different way than finite sets. Consider the set $X = \{a, b, c\}$. We can order X in $3! = 6$ different ways: a, b, c or b, a, c or c, b, a, etc. But other than specific labels these ordering all have the same abstract structure: first item, second item, third item. This is true for any finite set. If such a set is ordered, we will have: first item, second item, $\ldots$, last item.

For infinite sets this is no longer true. Consider the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. We can order this set in many distinct ways. For example, the standard order $0, 1, 2, \ldots$ behaves differently than the ordering $\ldots, 5, 3, 1, 0, 2, 4, \ldots$ (here we list the odd numbers backwards then followed by the even numbers). The first ordering has a first but not a last element while the other ordering has neither a first nor a last element. But how about $0, 2, 4, \ldots, 1, 3, 5, \ldots$ (the even numbers followed by the odd numbers)? This is quite different from our first (natural) ordering. Notice that in the natural ordering 0 is the only element not *immediately* preceded by any other element. However, in this new ordering both 0 and 1 are not *immediately* preceded by any other element. We could also consider $1, 2, 3, \ldots, 0$ (all positive numbers then followed by zero). This ordering has both a first and last element. So does $2, 3, 4, \ldots, 0, 1$ (all numbers two or larger then followed by zero and one) but this time the final element has an immediate predecessor. These examples should illustrate that there are infinitely many ways to organize the order structure of an infinite set. Let us get a little more technical.

Definition 3.1. *Consider a set X and define a relation $\leq$ on X so that for any two elements $a, b \in X$ we either have $a \leq b$ or not. Assume the following properties hold (for any $a, b, c \in X$):*

Reflexive: *We always have $a \leq a$.*

Antisymmetric: *If $a \leq b$ and $b \leq a$, then $a = b$*

Transitive: *If $a \le b$ and $b \le c$, then $a \le c$.*

Such a relation, $\le$, is called a partial order *on X (X is a "poset").*

If in addition, we have that for every $a, b \in X$ either $a \le b$ or $b \le a$, then we call the relation, $\le$, a total order *on X.*

From this we fix related notations as follows: $a < b$ means $a \le b$ but $a \ne b$ and $a \ge b$ means $b \le a$ etc. As a consequence, if X is totally ordered by $\le$, then for all $a, b \in X$ exactly one of the following holds: $a < b$, $a = b$, or $a > b$ (trichotomy).

Our usual $\le$ on $\mathbb{R}$ is a total order. This is also true for $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$. On the other hand, notice that for sets A, B, and C we always have that $A \subseteq A$ (reflexive), if $A \subseteq B$ and $B \subseteq A$ then $A = B$ (antisymmetric), and if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ (transitive). This means that given a collection of sets, the subset relation, $\subseteq$, partially orders that collection of sets. But generally this is not a total order. For example, consider $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ (the power set of $\{1,2\}$). Notice that both $\{1\} \not\subseteq \{2\}$ and $\{2\} \not\subseteq \{1\}$, so we fail to have a total order. Before getting to ordinals, we need one more ordering concept.

Definition 3.2. *Let X be totally ordered by $\le$. We say that X is* well ordered *by $\le$ if every nonempty subset of X has a least element: if $A \subseteq X$ and $A \ne \emptyset$, there is some $a \in A$ such that for all $x \in A$ we have $a \le x$.*

Notice that while $\mathbb{R}$ and $\mathbb{Z}$ are totally ordered, they are not well ordered (they have no least elements). On the other hand, $\mathbb{N} = \{0, 1, 2, \dots\}$ is, not just totally ordered but is in fact, well ordered by $\le$. In fact, this is usually referred to as the *Well Ordering Principle*. This property of the natural numbers is equivalent to the principle of mathematical induction (each principle easily implies the other) [Birkhoff, et al., 1967].

At this point the Axiom of Choice enters the picture. Loosely speaking, the Axiom of Choice states that one can select an element from each set in a (potentially infinite) collection of nonempty sets. More formally, if A_i (where $i \in I$ for some index set I) is a collection of nonempty sets, the Axiom of Choice asserts the existence of a function $f : I \to \bigcup\limits_{i \in I} A_i$ where for each $i \in I$ we have $f(i) \in A_i$. This can be restated as, a cartesian product of nonempty sets $\times_{i \in I} A_i$ is nonempty. The Axiom of Choice is often presented as a statement called Zorn's Lemma. It is also equivalent to statements that look rather harmless like: every vector space has a basis [Mendelson, 1964] and [Birkhoff, et al., 1967].

Most relevant to us, the Axiom of Choice is equivalent to stating that every set has a well ordering. For countable sets, this is fairly obvious. A countable set can be put into a one-to-one correspondence with a subset of $\mathbb{N}$. We can then use that correspondence to transport the counting numbers' ordering to our countable set. Things get weird when we turn our attention to larger sets. Consider the real numbers. While $\mathbb{R}$ is totally ordered by $\le$, this is not a well ordering (there is no minimum real number).[4]

We are now ready to build the collection of ordinal numbers. (As previously stated, some collections are too large to be sets. The ordinals form such a collection.) First, following John

[4]This cannot be fixed by simply moving one number out of place. In fact, every open interval, $I = (a, b)$, is a nonempty subset of $\mathbb{R}$, but none of these intervals contains a minimum element since $a \notin I$. Still, the Axiom of Choice states that $\mathbb{R}$ *can be* well ordered. It just turns out that such an order cannot be written down in a concrete way. In fact, Paul Cohen showed that it is consistent with Zermelo-Fraenkel set theory (i.e., standard set theory without the Axiom of Choice) that there is no well ordering on the reals [Kanamori, 2008]!

von Neumann, we will build a concrete model of the natural numbers from set theory. Let $0 = \emptyset$. We are not saying that secretly the number zero is the empty set. We are merely using the empty set as a model for how we want zero to work. Next, let $1 = \{0\} = \{\emptyset\}$, so 1 is the set containing the empty set.[5] Next, let $2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$. In general, let $n + 1 = n \cup \{n\}$. For example, $4 = 3+1 = 3\cup\{3\} = \{0, 1, 2\}\cup\{3\} = \{0, 1, 2, 3\}$. In other words, $n = \{0, 1, \ldots, n-1\}$. It is important to note that we are now employing n as both a ordinal number and as defining a set. This is a neat trick. We've built our numbers from literally nothing!

This construction has several nice features. First, we have that $m \leq n$ if and only if $m \subseteq n$ (as sets). Also, $n \cup m = \max\{n, m\}$ and $n \cap m = \min\{n, m\}$. Finally, notice that $\text{card}(n) = n$. This construction gives a collection of sets which (after defining addition and multiplication) satisfy Giuseppe Peano's axioms describing the natural numbers.

We can now create a new ordinal from an old one by the process of creating a successor: $s + 1 = s \cup \{s\}$. We can also create an ordinal by unioning a (potentially infinite) collection of ordinals. If this is a finite union of ordinals, we have $n_1\cup\cdots\cup n_k = \max\{n_1, \ldots, n_k\}$. Notice that $0\cup1\cup2\cup\cdots = \mathbb{N}$. Thus $\mathbb{N}$ is an ordinal. To indicate we are thinking of $\mathbb{N}$ not as the set of natural numbers but as the first infinite ordinal, we write $\omega = \mathbb{N}$, where ω represents the set of natural numbers together with order structure. Then $\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, \ldots, \omega\}$. Likewise, we can create $\omega + 1$, $\omega + 2$, etc. We even have $\omega \cdot 2 = \omega + \omega = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\}$. Notice that $\omega + \omega$ has the same order structure as one of our motivating ordering examples: $0, 2, 4, \ldots, 1, 3, 5, \ldots$ (all even numbers followed by all odd numbers).

We should distinguish between two kinds of ordinals. A *successor ordinal* is an ordinal, β, of the form $\beta = \kappa + 1$ for some ordinal κ. Ordinals such as 5 and 123 and $\omega + 2$ are successor ordinals since they are immediate successors of some other ordinal: $5 = 4 + 1$, $123 = 122 + 1$, $\omega + 2 = (\omega + 1) + 1$. If an ordinal is not a successor, it is called a *limit ordinal*; ω and $\omega+\omega$ are limit ordinals. We cannot find some κ such that $\kappa + 1 = \omega$ because κ would have to be ω with the "last" natural number removed. Likewise, $\omega + \omega \neq \kappa + 1$ for any κ. We only get ordinals like ω by unioning up an infinite collection of ordinals: $\omega = \bigcup_{n\in\mathbb{N}} n$; ω cannot be the result of performing a "successor" operation on some ordinal.

Therefore, all ordinals come from the operations of successor and unioning ordinals. In fact, given a well ordered set X, there is a unique ordinal κ and invertible function $f : X \to \kappa$ such that for all $a, b \in X$ we have $a \leq b$ (in X) if and only if $f(a) \leq f(b)$ (in κ). Thus, every well ordered set can be associated with a unique ordinal number. This means that each (well ordered) order structure is modeled by a unique ordinal. For example, $3 = \{0, 1, 2\}$ models the order structure of "first, second, last". We can now use this understanding to consider some arithmetic in the ordinals.

We begin with addition. Take two ordinals, α and β. Create a new set $\alpha \cup \beta'$ where β' is a disjoint copy of β (just take each element of β and add a prime to make it different). Give $\alpha\cup\beta'$ the natural ordering where we use α's ordering for elements drawn from α and β's ordering for elements from β' and declare that everything in α comes before everything in β. For example, $2 \cup 3'$ would be ordered: $0 < 1 < 0' < 1' < 2'$. Finally, it can be shown that $\alpha \cup \beta'$ with this ordering is well ordered and thus corresponds to a unique ordinal number. We define $\alpha + \beta$ to be that ordinal. For example, $2 + 3$ is the ordinal associated with $0 < 1 < 0' < 1' < 2'$ which corresponds to the order structure $0 < 1 < 2 < 3 < 4$. Thus $2 + 3 = 5$. We finally know why $1 + 1 = 2$!

[5] This often confuses students: $\emptyset = \{\}$ is empty while $\{\emptyset\} = \{\{\}\}$ is not, since it contains one thing: $\emptyset$.

Here's where things get interesting. Consider $3+\omega$. Thus would be $0 < 1 < 2 < 0' < 1' < 2' < 3' < \cdots$ which is the same order structure as ω. Therefore, $3+\omega = \omega$. But $\omega+3$ is ordered as $0 < 1 < 2 < \cdots < \omega < \omega+1 < \omega+2$, so $\omega+3 \neq \omega$. This means that ordinal addition is not commutative: $3+\omega \neq \omega+3$.

Next, we multiply. Again, let α and β be ordinals. Recall that each ordinal number is a set, so we can consider the Cartesian product $\beta \times \alpha$. We give $\beta \times \alpha$ are ordering as follows: $(b_1, a_1) \leq (b_2, a_2)$ if either $b_1 \leq b_2$ (in β) or $b_1 = b_2$ and $a_1 \leq a_2$ (in α). In other words, we order lexicographically. Again, $\beta \times \alpha$ is well ordered with such an ordering and thus is associated with a unique ordinal. We define $\alpha \times \beta$ to be that ordinal.

For example, consider $2 \cdot 3$. The Cartesian product $3 \times 2 = \{0,1,2\} \times \{0,1\}$ is ordered as follows: $(0,0) < (0,1) < (1,0) < (1,1) < (2,0) < (2,1)$. This has the same order structure as $0 < 1 < 2 < 3 < 4 < 5$ (i.e., 6) thus $2 \cdot 3 = 6$.

The product, $2 \cdot \omega$ has the structure determined by the set $\omega \times 2$ ordered as $(0,0) < (0,1) < (1,0) < (1,1) < (2,0) < \cdots$ which is the same order type as ω (so that $2 \cdot \omega = \omega$). On the other hand, $\omega \cdot 2$ corresponds to the set $2 \times \omega$ ordered as $(0,0) < (0,1) < (0,2) < \cdots < (1,0) < (1,1) < (1,2) < \cdots$. This means $\omega \cdot 2 = \omega + \omega$. In general, think of $\alpha \cdot \beta$ as β consecutive copies of α. Thus ω copies of 2 just yields ω but 2 copies of ω yields $\omega+\omega$. Since $\omega = 2 \cdot \omega \neq \omega \cdot 2 = \omega + \omega$, ordinal multiplication is also noncommutative.

Notice also, that while we have distributivity on the right: $\omega \cdot 2 = \omega \cdot (1+1) = \omega \cdot 1 + \omega \cdot 1 = \omega + \omega$, the distributive law does not work on the left: $\omega = 2 \cdot \omega = (1+1) \cdot \omega \neq 1 \cdot \omega + 1 \cdot \omega = \omega + \omega$. While some familiar rules work, like $\alpha \cdot 1 = \alpha = 1 \cdot \alpha$ and associativity, others rules fail. When adding and multiplying ordinals, we need to proceed with caution.

There is another approach to ordinal arithmetic, namely recursion, which mathematicians in this field commonly use. For addition, fix the base case: $\alpha + 0 = \alpha$. For successors like $\beta+1$, define $\alpha + (\beta+1)$ as the successor of $\alpha+\beta$ (i.e., $\alpha + (\beta+1) = (\alpha+\beta)+1$). Finally, for limit ordinals such as β, define $\alpha + \beta = \bigcup\limits_{\gamma \in \beta} (\alpha+\gamma)$. With addition established, multiplication is then built in a similar way from addition and recursion. Base case: $\alpha \cdot 0 = 0$. Successors: $\alpha \cdot (\beta+1) = \alpha \cdot \beta + \alpha$. Limit ordinals: $\alpha \cdot \beta = \bigcup\limits_{\gamma \in \beta} \alpha \cdot \gamma$. One can also define exponentiation from multiplication and recursion. Base case: $\alpha^0 = 1$. Successors: $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$. Limit ordinals: $\alpha^\beta = \bigcup\limits_{\gamma \in \beta} \alpha^\gamma$.

We could have defined exponentiation, α^β, by placing a lexicographic order certain order on a set of functions from β to α (functions of "finite support") and associating with an ordinal. However, let's not concern ourselves with this interpretation. Exponentiation like addition and multiplication can be weird. For example, $(\omega \cdot 2)^2 = (\omega \cdot 2) \cdot (\omega \cdot 2) = \omega \cdot (2 \cdot \omega) \cdot 2 = \omega \cdot \omega \cdot 2 = \omega^2 \cdot 2 = \omega^2 + \omega^2$. On the other hand, $\omega^2 \cdot 2^2 = \omega^2 \cdot 4 = \omega^2 + \omega^2 + \omega^2 + \omega^2$ which is strictly bigger than $\omega^2 + \omega^2$. Thus $(\omega \cdot 2)^2 \neq \omega^2 \cdot 2^2$. As one last bizarre example, we state without proof that $2^\omega = \omega$. Ordinal arithmetic can be very strange.

4 Connecting Ordinals with Cardinals

With cardinal and ordinal numbers in hand, we can now give an existence to our currently disembodied notion of "card(X)". We do so by connecting the two types of numbers.

It turns out that the collection of all ordinals is well ordered in the sense that any nonempty subcollection of ordinals contains a least ordinal (remember that ordinals are ordered by containment: $\alpha \leq \beta$ if and only if $\alpha \subseteq \beta$). If we consider all of the ordinals of a particular cardinality this is a non-empty collection, thus there must be a smallest such ordinal (smallest in the ordinal sense).[6] We call such an ordinal a *cardinal number*. There are ordinals of every cardinality, so given a set X, there is at least one ordinal α such that $\text{card}(X) = \text{card}(\alpha)$. Consider all of the ordinals with cardinality $\text{card}(X)$ and suppose that α comes first (to be clear, α is the first ordinal in the ordinal ordering whose cardinality matches that of X), then we say $\text{card}(X) = \alpha$.

This can be quite confusing mostly because when sets are finite, there is only one ordinal per cardinality. For example, the only ordinal with three elements is 3 itself! We write $\text{card}(\{a, b, c\}) = 3$. On the other hand, $\omega + 5$, ω^2, $\omega + \omega$ are all countable ordinals. But we also know that each collection of ordinals of the same cardinality must have a least element. For instance, while ω, $\omega + 17$, $\omega \cdot 3$, and ω^2 all have then same cardinality, as ordinals, $\omega < \omega + 17 < \omega \cdot 3 < \omega^2$. The smallest countable ordinal is ω itself thus $\text{card}(\omega) = \text{card}(\omega + 17) = \text{card}(\omega \cdot 3) = \text{card}(\omega^2) = \omega$.

When thinking of $\mathbb{N} = \omega = \{0, 1, 2, \ldots\}$ as a cardinal number, we call it $\aleph_0$. Our three labels $\mathbb{N}$, ω, and $\aleph_0$ represent the same collection of objects, but signify a different structure being considered (i.e., set versus ordinal versus cardinal). So $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Q}) = \aleph_0$ (this is countable infinity). The first uncountable ordinal (thought of as a cardinal) is called $\aleph_1$.

Cardinal arithmetic is much easier than ordinal arithmetic. We define addition as follows: $\text{card}(A) + \text{card}(B)$ is the cardinality of the disjoint union of A and B. It turns out that the cardinality of the union of two disjoint sets is the same as the larger of the two sets (as long as at least one of the sets is infinite), so for infinite cardinals $\text{card}(A) + \text{card}(B) = \max\{\text{card}(A), \text{card}(B)\}$. Thus $\aleph_1 + 5$, $\aleph_1 + \aleph_0$, and $\aleph_1 + \aleph_1$ are all just $\aleph_1$.

Cardinal multiplication is also easy. We define $\text{card}(A) \cdot \text{card}(B) = \text{card}(A \times B)$ (the cardinality of the Cartesian product of A and B). Again as along as at least one of our sets is infinite (and the other isn't empty) we have $\text{card}(A) \cdot \text{card}(B) = \max\{\text{card}(A), \text{card}(B)\}$ since in the context of infinite sets the Cartesian product's cardinality is just the cardinality of the bigger set. For example, $2\aleph_0$ and $\aleph_0 \cdot \aleph_0$ are both just $\aleph_0$. In summary, for nonzero cardinals α and β where at least one is infinite, we have that $\alpha + \beta = \alpha \cdot \beta = \max\{\alpha, \beta\}$.

Finally, exponentiation is also simple to define: $\text{card}(A)^{\text{card}(B)} = \text{card}(A^B)$ where A^B is the set of all functions $f : B \to A$ (functions from B to A).

Consider a set X and let $A \subseteq X$. We can define a characteristic function $\chi_A : X \to 2$ by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. Thus χ_A just tells us if an element of X belongs to A or not (think 1 means yes and 0 means no). Conversely, given a function $f : X \to 2$ we can let $A = \{x \in X \mid f(x) = 1\}$. In fact, if so, we have that $f = \chi_A$. So we can see that every subset of X is uniquely associated with a function from X to 2 (i.e., an element of 2^X). For instance, consider $\mathcal{P}(\{a, b, c\})$. Then the subset $A = \{a, c\}$ corresponds with the characteristic function $\chi_A(a) = \chi_A(c) = 1$ and $\chi_A(b) = 0$. Conversely, if $f : \{a, b, c\} \to \{0, 1\}$ is the function $f(a) = 1$ and $f(b) = f(c) = 0$ then $f = \chi_B$ where $B = \{a\}$. The function $g(a) = g(b) = g(c) = 0$ is the characteristic function of the empty set: $g = \chi_\emptyset$. The $2^3 = 8$ functions from $\{a, b, c\}$ to

[6]Note that we are considering two kinds of smallness/order. We order ordinals one way and cardinals another. It turns out that order on cardinal numbers is a sampling of the order on ordinals (all cardinal numbers are ordinal numbers but not vice-versa). If we consider cardinalities of ordinals, we see that $\alpha \leq \beta$ (as ordinals) implies $\text{card}(\alpha) \leq \text{card}(\beta)$. But $\text{card}(\alpha) = \text{card}(\beta)$ does not mean $\alpha = \beta$ (like we're used to for finite ordinals). For example, $\omega + 1$ and ω have equal cardinalities but are not the same ordinal.

$\{0,1\}$ correspond to the 2^3 subsets of $\{a,b,c\}$. Therefore, $\mathrm{card}(\mathcal{P}(X)) = \mathrm{card}(2^X) = 2^{\mathrm{card}(X)}$. Recalling Cantor's theorem, we see that $\mathrm{card}(X) < 2^{\mathrm{card}(X)}$. We can also now see that $\mathrm{card}(\mathbb{R}) = \mathrm{card}(\mathcal{P}(\mathbb{N})) = 2^{\aleph_0}$ (continuum). Notice that $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} \cdots$.[7] Indeed, therefore, there is an infinite number of infinities! Thanks, again, Buzz Lightyear for getting this right and communicating such so artfully.

Just as with addition and multiplication, exponentiation carries some oddities. For example, if we know that $A \subseteq B$, then there are at least as many functions from A to C as there are from B to C. Therefore, if $\alpha \leq \beta$, then $\alpha^\gamma \leq \beta^\gamma$. This implies that $2^{\aleph_0} \leq 3^{\aleph_0} \leq (\aleph_0)^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0}$. However, $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$ because $\aleph_0 \cdot \aleph_0 = \aleph_0$, since maximum computes multiplication.[8] Therefore, our whole chain of inequalities must actually be equalities! This means that:

$$2^{\aleph_0} = 3^{\aleph_0} = \aleph_0^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0}.$$

We will let the reader decide if this mathematical statement is weird or wonderful.

Since the real numbers are uncountable, $\aleph_1 \leq 2^{\aleph_0}$ (recall that $\aleph_1$ is the smallest uncountable cardinal). It is natural to ask if $\aleph_1 = 2^{\aleph_0}$? This is known as the Continuum Hypothesis (CH). A more wide sweeping conjecture came to be known as the Generalized Continuum Hypothesis (GCH). This asserts that $2^{\aleph_n} = \aleph_{n+1}$ for all ordinals n where $\aleph_{n+1}$ is next cardinal after $\aleph_n$. In other words, while Cantor tells us that the power set of X is larger than X, GCH states that there are no sizes in between. If CH were true, $2^{\aleph_0} = \aleph_1$, there would be no "sizes of infinity" between countable (i.e., the size of the natural numbers) and continuum (i.e., the size of the real numbers). If GCH were true, we would know exactly how cardinal exponentiation works (each exponential just moves us up to the next cardinal).

David Hilbert in his famous address to the International Congress of Mathematicians in 1900 gave a list of important open problems. The Continuum Hypothesis was the very first problem on his list. It came as a great surprise to mathematicians that CH cannot be proven or disproven within the context of Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC, i.e., standard set theory). In particular, in 1940 Kurt Gödel showed inside of ZFC one cannot disprove CH. On the other hand, in the 1963 Paul Cohen showed that inside ZFC one cannot prove CH either! In other words, either CH or its negation could be added to ZFC as a new and independent axiom. So in the context of standard set theory, we have no hope of truly understanding cardinal exponentiation – or *exactly* how big the real numbers really are!

5 Limiting

Notice that in the entire text above regarding cardinal and and ordinal numbers – even when representing countable and uncountable infinite notions – the symbol ∞ was never used. This is because ∞ provides a third context for infinity.

Students informally encounter ∞ as early as in elementary grades. They then encounter it more formally in precalculus and calculus in the context of limits. Note that, rather than calculating $f(\infty)$, we typically only claim to approach ∞ through a limit (i.e., $\lim\limits_{x\to\infty} f(x)$) rather

[7] Some authors use $\beth_n$ (Hebrew letter Beth) to represent the n^{th} powerset of $\mathbb{N}$ so $\beth_0 = \aleph_0 = \mathrm{card}(\mathbb{N})$, $\beth_1 = 2^{\aleph_0} = \mathrm{card}(\mathcal{P}(\mathbb{N}))$, $\beth_2 = 2^{\beth_1} = 2^{2^{\aleph_0}}$ etc.

[8] Functions $\aleph_0 \to (\aleph_0 \to 2)$ (functions from $\aleph_0$ to functions from $\aleph_0$ to 2) are essentially just functions $\aleph_0 \times \aleph_0 \to 2$ (functions from ordered pairs from $\aleph_0 \times \aleph_0$ to 2) essentially: $f(x,y) = f_x(y)$. Thus, $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0}$.

than actually hit ∞. While not totally unrelated, this notion of infinity is not the same as ω, $\aleph_0$, $2^{\aleph_0}$, or $\aleph_1$.

In many ways, ∞ denotes a location, a place we are heading towards, rather than a size. Often instructors and authors emphasize that we can approach but never get to ∞. As discussed in [Cook et al., 2019], this does not have to be the case. One can define both ∞ and/or $-\infty$ to be an extended kind of real number and then consider either one- or two-point compactifications of the real numbers: $\mathbb{R} \cup \{\infty\}$ or $\mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$. Here we can treat infinity much like any other real number and make sense out of arithmetic operations and function evaluations as long as we are sufficiently careful.

As with cardinal and ordinal numbers, some arithmetic is afforded us in respect to ∞. For instance, for $a \in \mathbb{R}$ and $a > 0$: $\infty + a \cdot \infty = \infty$, $\infty^a = \infty$, and $\infty/a = \infty$. However, $\infty - \infty$ and ∞/∞ are ill defined.

The brevity of our consideration of ∞ in this context is due to the respective commonality with which it is encountered by students and our discussion in [Cook et al., 2019].

6 Investigations

In this section, we provide some investigations appropriate to high school and undergraduate students who wish to challenge themselves and understand more of the concepts of infinity.

1. For both cardinal and ordinal numbers and for both the operations of addition and multiplication, determine which properties hold and provide counter examples if they fail: commutativity, associativity, and distributivity.

2. Explain why the collection of polynomials with integer coefficients has a cardinality of $\aleph_0$. Use this to explain why the set of Algebraic Numbers must also have a cardinality of $\aleph_0$.

3. Give examples of sets which are: (a) totally ordered; (b) well ordered; (c) totally ordered but not well ordered.

4. Find at least five resources discussing the Axiom of Choice and summarize in your own words.

5. In the paper, we wrote: "We can place an equivalence relation on the collection of ordinals as follows: $\alpha \sim \beta$ if and only if $\text{card}(\alpha) = \text{card}(\beta)$... Every finite ordinal is a cardinal and distinct finite ordinals have distinct cardinalities... Each equivalence class of ordinals of the same cardinality must have a least element... We call the smallest such ordinal a cardinal number." Rewrite these sentences in your own words with your own examples.

7 Conclusion

We have considered three contexts for the notion of infinity: cardinal number ($\aleph_0$ and $2^{\aleph_0}$); ordinal number (ω and $\omega + \omega$); and as a destination (∞). Most high school students have only had experience with the latter. We hope that this brief consideration has opened new vistas regarding infinity and stimulates interest in these wonderful notions.

References

[Birkhoff, et al., 1967] Birkhoff, G., S. MacLane. (1967) *Algebra*. MacMillan.

[Cook et al., 2019] Cook W., M. Bossé. *Infinity ONE: Rational Functions*, ???

[Halmos, 2011] Halmos, P. (2001) *Naive Set Theory*. Springer.

[Kanamori, 2008] Kanamori, A. "Cohen and set theory". *Bulletin of Symbolic Logic*, Sept. 2008, Vol. 14, Number 3.

[Mendelson, 1964] Mendelson, E. (1964) *Introduction to Mathematical Logic*. D. Van Nostrand.

[Roitman, 1990] Roitman, J. (1990) *Introduction to Modern Set Theory*. Wiley.