Supplemental Notes: Line Integrals

Notation:

- Let C be an oriented curve parameterized by $\mathbf{X}(t) = (x(t), y(t), z(t))$ where $a \leq t \leq b$.
- $-C$ denotes the curve C with its orientation reversed.
  $C_1 + C_2$ means take curve C_1 and curve C_2 and put them together.
- Let Let $\mathbf{F} = (P, Q, R)$ be a vector field defined on (and around) C.
- Define $s(t) = \displaystyle\int_a^t |\mathbf{X}'(u)|\, du$. This is the *arc length* function. *Note:* $s(b)$ is the arc length of C.
- Recall that $\mathbf{T}(t) = \dfrac{\mathbf{X}'(t)}{|\mathbf{X}'(t)|}$ is the *unit Tangent* function.
- Also, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ is the *unit Normal* and $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is the *Binormal.*

Differentiate the arc length function and get $\dfrac{ds}{dt} = |\mathbf{X}'(t)|$ (by the fundamental theorem of calculus). Treating ds and dt like formal variables, we get "$ds = |\mathbf{X}'(t)|\, dt$". Following the same convention we make the following notational definitions:

- $ds = |\mathbf{X}'(t)|\, dt = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$
- $x = x(t)$, $y = y(t)$, and $z = z(t)$ so $dx = x'(t)dt$, $dy = y'(t)dt$ and $dz = z'(t)dt$
- $\mathbf{X}(t) = (x(t), y(t), z(t))$ and so $d\mathbf{X} = \mathbf{X}'(t)\, dt = (x'(t), y'(t), z'(t))\, dt = (dx, dy, dz)$
- $\mathbf{F} = \mathbf{F}(\mathbf{X}(t)) = (P(x(t), y(t), z(t)), Q(x(t), y(t), z(t)), R(x(t), y(t), z(t)))$

So we can define line integrals:

- $\displaystyle\int_C g(x, y, z)\, ds = \int_a^b g(\mathbf{X}(t))\, |\mathbf{X}'(t)|\, dt = \int_a^b = g(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$
- $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{X} = \int_C P\, dx + Q\, dy + R\, dz = \int_a^b \mathbf{F}(\mathbf{X}(t)) \cdot \mathbf{X}'(t)\, dt$
  $\displaystyle= \int_a^b P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)\, dt$

Note: $\displaystyle\int_C 1\, ds =$ Arc Length of C

Remember that both types of line integrals are **independent of parameterization** (as long as the orientation of C is preserved) – that is – using a different parameterization will not change the answer. Also, if the line integral is done with respect to arc length, then orientation does not matter. Whereas, with the other type of line integral, reversing the orientation negates the result.

- $\displaystyle\int_{-C} g(x, y, z)\, ds = \int_C g(x, y, z)\, ds$
- $\displaystyle\int_{-C} \mathbf{F} \cdot d\mathbf{X} = -\int_C \mathbf{F} \cdot d\mathbf{X}$

Work and Flux

Suppose $\mathbf{F}$ is a force field (like the force due to gravity). We know that work is force dotted with a displacement vector (when force is constant anyway). Consider a particle being moved along an oriented curve C. If we focus on a small enough portion of our curve the force should be (approximately) constant. So the work to move the particle along this small portion of the curve will be $\mathbf{F} \cdot \mathbf{T}\,\Delta s$ where $\mathbf{T}$ is the unit tangent and Δs is the length of this part of the curve. Why $\mathbf{T}\,\Delta s$? Well, since we are focusing on such a small part of our curve, the curve should look (approximately) like a straight line, so there is no difference between our curve and it's tangent – "moving along the curve = moving along the tangent" (approximately anyway). Now adding up the work moving along all small pieces of our curve we get $\Sigma \mathbf{F} \cdot \mathbf{T}\,\Delta s$ which translated to the world of integrals is...

$$\text{Work} \;=\; \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot \frac{\mathbf{X}'}{|\mathbf{X}'|}\,|\mathbf{X}'|\, dt = \int_C \mathbf{F} \cdot d\mathbf{X}$$

Seeing the appearance of $\mathbf{T}$ in the integral above might lead you to ask, "What happens if we use $\mathbf{N}$ (the unit normal) instead?" The following integral is called the **flux** of $\mathbf{F}$ across C (Note: we can't get rid of $\mathbf{N}$ like we did $\mathbf{T}$):

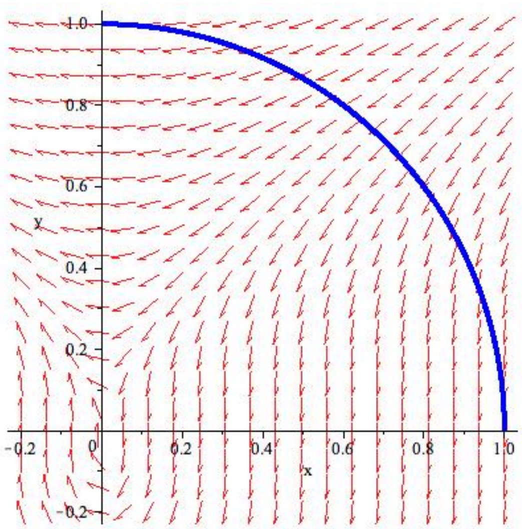
$$\text{Flux} \;=\; \int_C \mathbf{F} \cdot \mathbf{N}\, ds$$

Imagine that $\mathbf{F}$ is the velocity of some fluid crossing the curve C (pretend C is a filter or a net). Then $\mathbf{F} \cdot \mathbf{N}$ will grab the component of the velocity vector which is perpendicular to C. So it picks out the "part" of the fluid crossing C. Thus flux is measuring how much fluid is crossing over the curve C.

Example: See the book for examples involving work. Let's compute the flux of $\mathbf{F}(x,y) = (-3y^2, -2x)$ across C where C is the quarter of the unit circle in the first quadrant oriented counter-clockwise. C is parametrized by $\mathbf{X}(t) = (\cos(t), \sin(t))$ with $0 \leq t \leq \pi/2$ so that $\mathbf{X}'(t) = (-\sin(t), \cos(t)) = \mathbf{T}(t)$ since $|\mathbf{X}'(t)| = 1$. Next, $\mathbf{T}'(t) = (-\cos(t), -\sin(t))$ so $\mathbf{N}(t) = \dfrac{1}{|\mathbf{T}'(t)|}\mathbf{T}'(t) = (-\cos(t), -\sin(t))$.

$$\begin{aligned}\text{Flux} \;=\; \int_C \mathbf{F} \cdot \mathbf{N}\, ds \;&=\; \int_0^{\pi/2} (-3(\sin(t))^2, -2\cos(t)) \cdot (-\cos(t), -\sin(t))\, 1\, dt \\ &=\; \int_0^{\pi/2} 3(\sin(t))^2\cos(t) + 2\sin(t)\cos(t)\, dt \\ &=\; (\sin(t))^3 + (\sin(t))^2 \Big|_0^{\pi/2} = (1^3 + 1^2) - (0^3 + 0^2) = 2\end{aligned}$$

A plot of $\mathbf{F}$ (scaled down) and C.



Note: Since the curve C is oriented counter-clockwise, its normal vectors point inward and thus we get a positive flux.

Center of mass

Suppose we have a wire bent in the shape of the curve C and suppose this write has density $\rho(x,y,z)$ at each point (x,y,z) along the curve. Then if we focus on a little segment of the wire where the density is roughly constant, the mass of the segment of wire will be approximately $\rho(x_0,y_0,z_0)\Delta s$ where Δs is the length of this piece of the wire. So if we add up $\Sigma\rho\Delta s$ we should get the total mass of the wire (approximately anyway). Translating to the world of integrals we have...

- mass $= m = \displaystyle\int_C \rho(x,y,z)\,ds$
- Let $\bar{x} = \dfrac{1}{m}\displaystyle\int_C x\rho(x,y,z)\,ds$, $\quad \bar{y} = \dfrac{1}{m}\displaystyle\int_C y\rho(x,y,z)\,ds$, $\quad$ and $\quad \bar{z} = \dfrac{1}{m}\displaystyle\int_C z\rho(x,y,z)\,ds$

Where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are weighted averages of x, y, and z coordinates. We call $(\bar{x},\bar{y},\bar{z})$ the **center of mass** of the wire. If $\rho(x,y,z) = c =$ constant $\neq 0$, then $(\bar{x},\bar{y},\bar{z})$ is called the **centroid** of C.

Example: Let C be the helix parameterized by $\mathbf{X}(t) = (\cos(t), \sin(t), \sqrt{3}t)$ for $0 \leq t \leq 2\pi$. Then $\mathbf{X}'(t) = (-\sin(t), \cos(t), \sqrt{3})$ so $|\mathbf{X}'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (\sqrt{3})^2} = \sqrt{2}$. Thus $ds = 2\,dt$. The arc length of C is...

$$\int_C 1\,ds = \int_0^{2\pi} 2\,dt = 4\pi$$

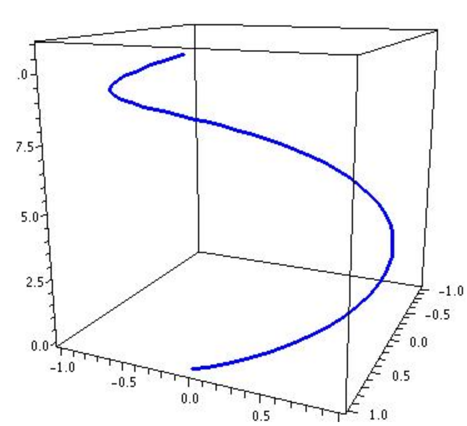
Let's compute the centroid of C. So let ρ be some non-zero constant, say $\rho = 1$ (the easiest constant to work with).

$$\bar{x} = \frac{1}{m}\int_C x\,ds = \frac{1}{4\pi}\int_0^{2\pi} \cos(t)2\,dt = 0$$

$$\bar{y} = \frac{1}{m}\int_C y\,ds = \frac{1}{4\pi}\int_0^{2\pi} \sin(t)2\,dt = 0$$

$$\bar{z} = \frac{1}{m}\int_C z\,ds = \frac{1}{4\pi}\int_0^{2\pi} \sqrt{3}t2\,dt = \frac{\sqrt{3}}{4\pi}\int_0^{2\pi} 2t\,dt = \frac{\sqrt{3}}{4\pi}t^2\Big|_0^{2\pi} = \frac{\sqrt{3}}{4\pi}(2\pi)^2 - 0 = \pi\sqrt{3} \approx 5.4414$$

Notice the helix's z-coordinates range from $z = 0$ to $z = \sqrt{3}\cdot 2\pi$ so $\bar{z}$ is exactly half way between 0 and $\sqrt{3}\cdot 2\pi$ (looking at a picture of this helix should convince you that this is the right answer).
Answer: $(\bar{x},\bar{y},\bar{z}) = (0, 0, \pi\sqrt{3})$

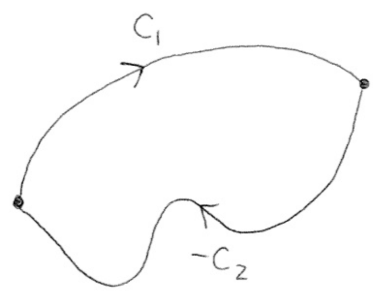


Conservative Vector Fields

Let $\mathbf{F}$ be a vector field defined on $\mathbb{R}^n$ and let R be some (open) subset of $\mathbb{R}^n$. Here's a few definitions:

- $\mathbf{F}$ is a **gradient vector field** if there exists a (scalar valued) function f such that $\mathbf{F} = \nabla f$. f is called a **potential** function.
- We say an integral $\int_C \mathbf{F} \cdot d\mathbf{X}$ is **path independent** if for any other curve C_2 with the same end points and orientation as C we have that $\int_C \mathbf{F} \cdot d\mathbf{X} = \int_{C_2} \mathbf{F} \cdot d\mathbf{X}$.
- $\mathbf{F}$ is a **conservative vector field** on R if $\int_C \mathbf{F} \cdot d\mathbf{X}$ is path independent for all curves C which lie inside the region R.

Let C and C_2 be curves with the same start and end points (and the same orientation).



The curve $C_1 - C_2$ is called a **simple closed curve**
Closed because it's start and end points match and
Simple because it does not cross itself.

Then assuming path independence we have...

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{X} = \int_{C_2} \mathbf{F} \cdot d\mathbf{X} \iff \int_{C_1} \mathbf{F} \cdot d\mathbf{X} - \int_{C_2} \mathbf{F} \cdot d\mathbf{X} = 0 \iff$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{X} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{X} = 0 \iff \int_{C_1 - C_2} \mathbf{F} \cdot d\mathbf{X} = 0$$

So requiring path independence is the same as requiring that all line integrals around closed loops turn out to be 0. This explains the name "conservative" since if $\mathbf{F}$ is a force field and so these line integrals compute work, then when a particle starts and ends in the same place no work has been done because energy was *conserved*.

A non-simple closed curve.

Recall our notation: C is a curve parameterized by $\mathbf{X}(t)$ where $a \leq t \leq b$. Let $\mathbf{A} = \mathbf{X}(a)$ (the starting point) and $\mathbf{B} = \mathbf{X}(b)$ (the ending point).

Theorem: (The Fundamental Theorem of Line Integrals)

$$\int_C \nabla f \cdot d\mathbf{X} = f(\mathbf{B}) - f(\mathbf{A})$$

proof:

$$\int_C \nabla f \cdot d\mathbf{X} = \int_a^b \nabla f(\mathbf{X}(t)) \cdot \mathbf{X}'(t)\, dt = \int_a^b (f \circ \mathbf{X})'(t)\, dt = (f \circ \mathbf{X})(t)\Big|_a^b = f(\mathbf{X}(b)) - f(\mathbf{X}(a)) = f(\mathbf{B}) - f(\mathbf{A})$$

The second equality is established using the chain rule and the third equality is the regular fundamental theorem of calculus. □

The fundamental theorem of line integrals states that line integrals involving gradient vector fields can be computed using the endpoints of curves alone! This says that line integrals involving gradient vector fields are *path independent*. Put another way...

Theorem: Gradient vector fields are conservative.

The converse of this theorem also holds, but it is a bit harder to show. Let R be a connected open subset of $\mathbb{R}^n$. By "connected" we mean that any two points in R are connected by a curve in R. "Open" means that R has "fuzzy" edges. Technically we mean that given any point in R there is a disk/ball surrounding that point (possibly with an extremely small radius) which lies entirely inside R.

Theorem: If $\mathbf{F}$ is conservative on an (open connected) region R, then $\mathbf{F}$ is a gradient vector field.

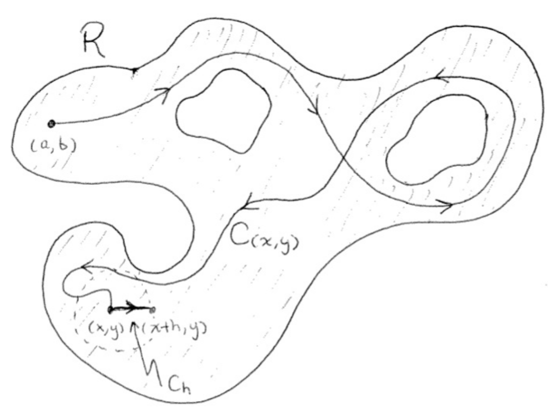
proof: We will prove this for a vector field $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ defined on an open connected subset of $\mathbb{R}^2$, but the theorem still holds in any dimension (the same proof works too – but the notation gets a bit messy).

First, fix a point (a,b) in R. Let (x,y) be some point in R and let $C_{(x,y)}$ denote some curve from (a,b) to (x,y). Note that such a curve exists since R is connected.

Let's define our potential potential function: $f(x,y) = \int_{C_{(x,y)}} \mathbf{F} \cdot d\mathbf{X}$. We should make the following **very important** note: since $\mathbf{F}$ is conservative, the value of our integral is independent of our choice of path. Thus any choice of $C_{(x,y)}$ will yield the same value $f(x,y)$. In other words, our function f is "well defined" on all of R.

We want to show that $\nabla f = \mathbf{F}$. So we need to show $f_x = P$ and $f_y = Q$. Let's focus on $f_x = P$.

First, some prep work. Fix some point (x,y) in R and let h be a real number small enough so that $(x+h,y)$ is in R as well (such numbers exist because R is open). Next, let $C_{(x+h,y)} = C_{(x,y)} + C_h$ where $C_{(x,y)}$ is any path from (a,b) to (x,y) and C_h is the line segment parameterized by $\mathbf{X}(t) = (t,y)$ where $x \leq t \leq x+h$. So $C_{(x+h,y)}$ is a path from (a,b) to (x,y) to $(x+h,y)$.



$$f(x+h,y) = \int_{C_{(x+h,y)}} \mathbf{F}\cdot d\mathbf{X} = \int_{C_{(x,y)}+C_h} \mathbf{F}\cdot d\mathbf{X} = \int_{C_{(x,y)}} \mathbf{F}\cdot d\mathbf{X} + \int_{C_h} \mathbf{F}\cdot d\mathbf{X} = f(x,y) + \int_{C_h} \mathbf{F}\cdot d\mathbf{X}$$

We will compute the integral over C_h by plugging in our parameterization. Remember $\mathbf{X}(t) = (t,y)$ so $\mathbf{X}'(t) = (1,0)$ (since y is fixed and t is our variable). Thus

$$f(x+h,y) - f(x,y) = \int_{C_h} \mathbf{F}\cdot d\mathbf{X} = \int_x^{x+h} (P(t,y),Q(t,y))\cdot(1,0)\,dt = \int_x^{x+h} P(t,y)\,dt$$

Let $g(x,y)$ be an antiderivative (integrating with respect to x) of $P(x,y)$ (so $g_x = P$). Then

$$f(x+h),y) - f(x,y) = \int_x^{x+h} P(t,y)\,dt = g(t,y)\Big|_x^{x+h} = g(x+h,y) - g(x,y)$$

Finally, we find that

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0} \frac{g(x+h,y)-g(x,y)}{h} = g_x(x,y) = P(x,y)$$

A similar proof shows that $f_y(x,y) = Q(x,y)$. Therefore, $\nabla f(x,y) = (f_x(x,y), f_y(x,y)) = (P(x,y),Q(x,y)) = \mathbf{F}(x,y)$ and thus $\mathbf{F}$ is a gradient vector field on R. $\square$

Thus, on an (open connected) region R, every gradient vector field is conservative and every conservative vector field is a gradient vector field. This leaves us with the question, "Given a vector field $\mathbf{F}$, how can we tell if $\mathbf{F}$ is gradient/conservative?" The following theorem helps us partially answer the question:

Theorem: Let $\mathbf{F} = (P,Q)$ and suppose P and Q have continuous first partials on some region R. If $\mathbf{F}$ is gradient on R, then $P_y = Q_x$. This also tells us that if $P_y \neq Q_x$ at some point in R then $\mathbf{F}$ cannot be gradient on R.

proof: Suppose that $\mathbf{F} = \nabla f$. Then $P = f_x$ and $Q = f_y$. Thus $P_y = f_{xy} = f_{yx} = Q_x$ (since we assumed that these partials are continuous Clairaut's theorem applies). $\square$

Next, we might ask, "What about vector fields in $\mathbb{R}^3$?" To discuss these we need to introduce the "curl" operator.

Definition: Given a vector field $\mathbf{F} = (P,Q,R)$, define $\text{curl}(\mathbf{F}) = \nabla\times\mathbf{F}$ as follows:

$$\text{curl}(\mathbf{F}) = \nabla\times\mathbf{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix}\vec{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{vmatrix}\vec{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}\vec{k} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

The notation indicates that you are (sort of) "crossing" the gradient operator with your vector field.

Quick Example: Let $\mathbf{F}(x,y,z) = (xyz, x^2y, y^2+z^2)$. Then $\text{curl}(\mathbf{F}) = \nabla\times\mathbf{F} = \ldots$

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & x^2y & y^2+z^2 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2+z^2 \end{vmatrix}\vec{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xyz & y^2+z^2 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xyz & x^2y \end{vmatrix}\vec{k} = (2y-0, xy-0, 2xy-xz)$$

An Important Computation: If f has continuous second partials, then $\text{curl}(\nabla f) = \nabla \times \nabla f = \mathbf{0}$.

$$\text{curl}(\nabla f) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_y & f_z \end{vmatrix} \vec{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ f_x & f_z \end{vmatrix} \vec{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ f_x & f_y \end{vmatrix} \vec{k} = (f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy}) = (0,0,0)$$

Since we assumed that f has continuous second partials, Clairaut's theorem applies (mixed partials are equal).

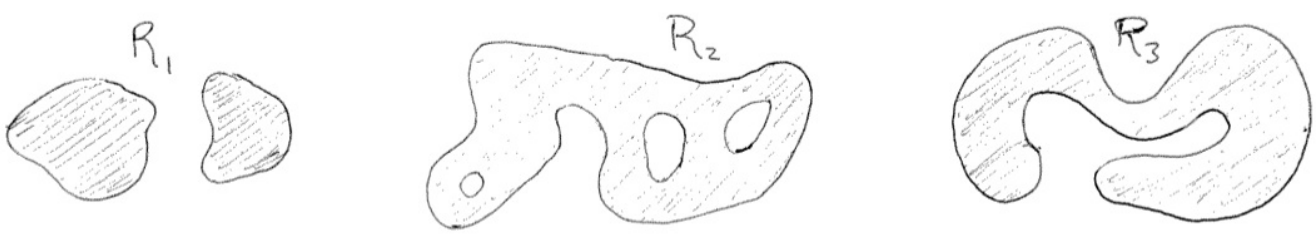
This computation shows us that...

Theorem: Let $\mathbf{F} = (P, Q, R)$ and suppose that P, Q, and R have continuous first partials on some region R. If $\mathbf{F}$ is a gradient vector field, then $\text{curl}(\mathbf{F}) = \mathbf{0}$ on R. This also tells us that if $\text{curl}(\mathbf{F}) \neq \mathbf{0}$ at some point in R, then $\mathbf{F}$ cannot be a gradient vector field on R.

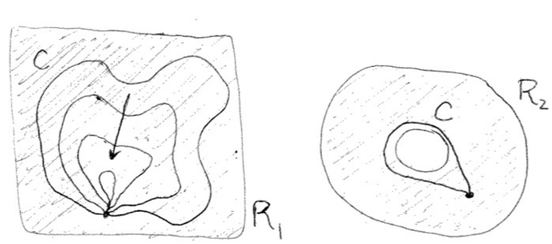
Note: If $\mathbf{F}(x, y, z) = (P(x, y), Q(x, y), 0)$ where P and Q do **not** depend on z, then we have: $\text{curl}(\mathbf{F}) = (0, 0, Q_x - P_y) = \mathbf{0}$ if and only if $P_y = Q_x$. So our 2-dimensional theorem is just a special case of the 3-dimensional theorem.

Our final question is, "Do the converses of this theorem and the last theorem hold?" The answer is YES on "simply" connected domains and possibly NO on other domains.

Definition: A region R of $\mathbb{R}^n$ is **simply connected** if R is connected (every two points in R can by joined by a path in R) and every closed path can be *contracted to a point.* Without getting too technical, the idea is: Given a closed path (one whose start and end points are the same), we can deform/stretch/contract it until there is nothing left. Essentially, we need R to be free of 1-dimensional holes.



R_1 is not connected, R_2 is connected but not simply, R_3 is simply connected

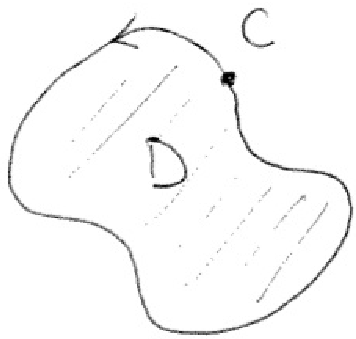


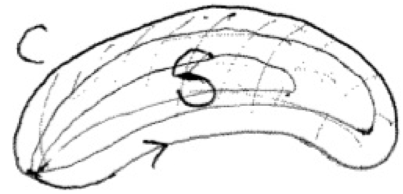
On the left, the curve C can be contracted to a point.
However, on the right, the curve C gets "stuck" on the hole in R_2.

Theorem: Let R be a simply connected open subset of $\mathbb{R}^3$ and suppose $\text{curl}(\mathbf{F}) = \mathbf{0}$ on R. Then $\mathbf{F}$ is a conservative/gradient vector field on R. If R is a simply connected open subset of $\mathbb{R}^2$ and $\mathbf{F} = (P, Q)$ where $P_y = Q_x$ on R, then $\mathbf{F}$ is a conservative/gradient vector field on R.

proof: The 2-dimensional version follows from Green's theorem and the 3-dimensional version follows from Stoke's theorem.

Accepting these theorems for the time being, here's the proof: Let C be a simple closed curve in R. If R is a subset of $\mathbb{R}^2$, then C bounds some region D. If R is a subset of $\mathbb{R}^3$, since R is simply connected, we can contract C to a point, while doing this C sweeps out a surface S in R.





D is the region bounded by C in $\mathbb{R}^2$, S is the surface swept out as C is contracted to a point in $\mathbb{R}^3$.

In 2-dimensions use Green's theorem: $$\int_C \mathbf{F} \cdot d\mathbf{X} = \iint_D Q_x - P_y \, dA = \iint_D 0 \, dA = 0$$

or

In 3-dimensions use Stoke's theorem: $$\int_C \mathbf{F} \cdot d\mathbf{X} = \iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

So we have (in either case) that $\mathbf{F}$ is conservative. $\square$

Example: We showed that for $\mathbf{F}(x, y, z) = (xyz, x^2y, y^2 + z^2)$, $\text{curl}(\mathbf{F}) = (2y, xy, x(2y - z)) \neq \mathbf{0}$. So this vector field is not conservative. Likewise, if $\mathbf{F}(x, y) = (x^2, xy)$ then $P_y = 0 \neq Q_x = y$ so $\mathbf{F}$ is not conservative.

Example: Let $\mathbf{F}(x, y) = (2x + y^2, e^y + 2xy)$. Then $P_y = 2y = Q_x$ so $\mathbf{F}$ is conservative (everywhere). Let's find a potential function for $\mathbf{F}$.

We need to find a function f so that $f_x = 2x + y^2$ and $f_y = e^y + 2xy$. Integrating both of these equations we find that $f(x, y) = x^2 + xy^2 + g(y)$ and $f(x, y) = e^y + xy^2 + h(x)$ (notice that our "arbitrary constants" need to be *functions* of the "other" variable). Putting these answers together we find that $f(x, y) = x^2 + xy^2 + e^y + C$ where C is any constant (this time an honest to goodness constant not a function). In particular, $\mathbf{F}(x, y) = \nabla f(x, y)$ where $f(x, y) = x^2 + xy^2 + e^y$ (choosing $C = 0$).

Example: Let $\mathbf{F}(x, y, z) = (2xyz + z, x^2z + 3y^2 + e^z, x^2y + ye^z + x)$. Then

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz + z & x^2z + 3y^2 + e^z & x^2y + ye^z + x \end{vmatrix} =$$

$$\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z + 3y^2 + e^z & x^2y + ye^z + x \end{vmatrix} \vec{\imath} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xyz + z & x^2y + ye^z + x \end{vmatrix} \vec{\jmath} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xyz + z & x^2z + 3y^2 + e^z \end{vmatrix} \vec{k}$$

$$= \big((x^2+e^z)-(x^2+e^z),(2xy+1)-(2xy+1),(2xz)-(2xz)\big) = (0,0,0)$$

Thus $\mathbf{F}$ is conservative. Again, let's find a potential function.

This time we need to find a function f so that $f_x = 2xyz + z$, $f_y = x^2z + 3y^2 + e^z$, and $f_z = x^2y + ye^z + x$. Integrating all three of these equations (again remember that the "constants" are really functions of the "other" variables), we get $f(x,y,z) = x^2yz + xz + g_1(y,z)$, $f(x,y,z) = x^2yz + y^3 + ye^z + g_2(x,z)$, and $f(x,y,z) = x^2yz + ye^z + xz + g_3(x,y)$. So putting this all together we have that $f(x,y,z) = x^2yz + xz + y^3 + ye^z + C$ where C is any constant. In particular, $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$ where $f(x,y,z) = x^2yz + xz + y^3 + ye^z$ (choosing $C=0$).

Example: Let $\mathbf{F}(x,y) = \left(\dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}\right)$. We should check to see if $\mathbf{F}$ is conservative. After some work we find that...

$$P_y = Q_x = \frac{y^2-x^2}{(x^2+y^2)^2}$$

So $\mathbf{F}$ is conser...**WAIT!** Notice that these partials are not continuous at the origin (they're not even defined there). We can only conclude that $\mathbf{F}$ is conservative on any simply connected domain which does not contain the origin. Notice that "simply connected" rules out the "domain" of $\mathbf{F}$ which is $\mathbb{R}^2 - \{(0,0\}$ (the whole plane with the origin deleted). The partials match on the entire domain of $\mathbf{F}$, the domain is connected, but it isn't *simply* connected. In fact, to demonstrate what can happen...

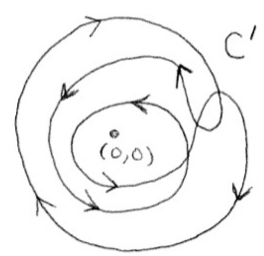
Let C be the unit circle oriented counter-clockwise. Parameterize C by $\mathbf{X}(t) = (\cos(t), \sin(t)$ where $0 \leq t \leq 2\pi$.

$$\int_C \mathbf{F}\cdot d\mathbf{X} = \int_0^{2\pi} \left(\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2(t)}\right)\cdot(-\sin(t),\cos(t))\,dt$$

$$= \int_0^{2\pi} \sin^2(t)+\cos^2(t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$$

Since our integral around the closed curve C is not 0, $\mathbf{F}$ is not conservative on $\mathbb{R}^2 - \{(0,0)\}$.

Note: Let C' be some closed curve. It can be shown that $\displaystyle\int_{C'} \mathbf{F}\cdot d\mathbf{X} = 2\pi(A-B)$ where A is the number of times C' wraps itself around the origin in a counter-clockwise fashion and B is the number of times that C' wraps itself around the origin in a clockwise fashion. $\displaystyle\frac{1}{2\pi}\int_{C'} \mathbf{F}\cdot d\mathbf{X}$ computes "winding numbers".



$$\int_{C'} \mathbf{F}\cdot d\mathbf{X} = 2\pi(2-1) = 2\pi$$