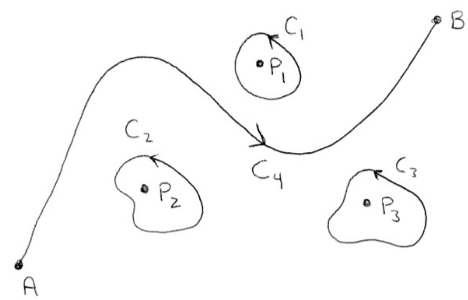
Supplemental Problems: Vector Calculus

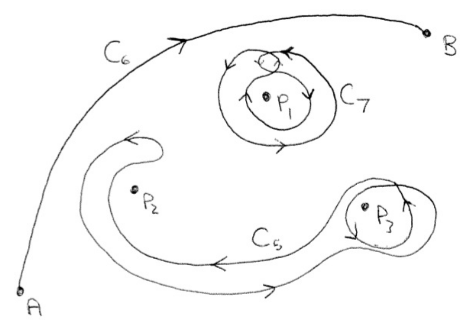
(Finding Potentials, Greens Theorem, Stokes Theorem, and The Divergence Theorem)

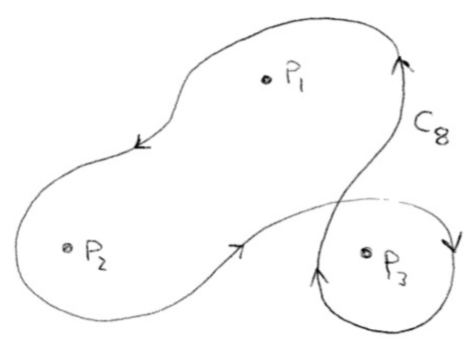
Determine if the following vector fields are conservative. For those which are conservative, find a potential function.

1. $\mathbf{F}(x,y) = \left(x^2+y^2, xy\right)$
2. $\mathbf{F}(x,y) = \left(2x+3x^2y^2+5, 2x^3y\right)$
3. $\mathbf{F}(x,y) = \left(e^x, ye^{-y^2}\right)$
4. $\mathbf{F}(x,y) = \left(e^{xy}, x^4y^3+y\right)$
5. $\mathbf{F}(x,y) = \left(e^x + \dfrac{y}{1+x^2}, \arctan(x)+(1+y)e^y\right)$
6. $\mathbf{F}(x,y,z) = (yz+y+1, xz+x+z, xy+y+1)$
7. $\mathbf{F}(x,y,z) = \left(2xyz+3x^2, x^2z+6z, 6y\right)$
8. $\mathbf{F}(x,y,z) = \left(yze^{xy}+2xz^2, xze^{xy}+7y^6+3y^2z^2, e^{xy}+2x^2z+(1+2z)e^{2z}+2y^3z\right)$

Let $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ be a vector field where $P_y = Q_x$ except at the points P_1, P_2, and P_3. Suppose that $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{X} = 1$, $\displaystyle\int_{C_2} \mathbf{F}\cdot d\mathbf{X} = 2$, $\displaystyle\int_{C_3} \mathbf{F}\cdot d\mathbf{X} = 3$, and $\displaystyle\int_{C_4} \mathbf{F}\cdot d\mathbf{X} = 4$ where C_1, C_2, $\ldots C_8$ are pictured to the right.







Compute:

9. $\displaystyle\int_{C_5} \mathbf{F}\cdot d\mathbf{X}$ [Answer: 6]

10. $\displaystyle\int_{C_6} \mathbf{F}\cdot d\mathbf{X}$ [Answer: 3]

11. $\displaystyle\int_{C_7} \mathbf{F}\cdot d\mathbf{X}$ [Answer: 0]

12. $\displaystyle\int_{C_8} \mathbf{F}\cdot d\mathbf{X}$ [Answer: 0]

Green's Theorem:

13. Let C be the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ (oriented counter-clockwise). Compute the line integral: $\int_C y^2\,dx + x^2\,dy$ two ways. First, compute the integral directly by parameterizing each side of the square. Then, compute the answer again using Green's Theorem.

14. Compute $\int_C \mathbf{F}\cdot d\mathbf{X}$ where $\mathbf{F}(x,y) = (y - \ln(x^2+y^2), 2\arctan(y/x))$ and C is the circle $(x-2)^2 + (y-3)^2 = 1$ oriented counter-clockwise. [Answer: $-\pi$]

15. Let C be the boundary of the part of the disk $x^2+y^2 \leq 16$ which lies in the first quadrant (oriented counter-clockwise). Compute $\int_C x^2y\,dx - y^2x\,dy$. [Answer: -32π]

16. Let C be the line segments from $(1,0)$ to $(1,1)$, $(1,1)$ to $(0,1)$, and $(0,1)$ to $(0,0)$. Compute $\int_C \sqrt{1+y^3}\,dx + (x^2+e^{-y^2})\,dy$.
Hint: Use Green's Theorem to replace C with the line segment from $(1,0)$ to $(0,0)$.

Surface & Flux Integrals:

17. Find the centroid of the upper-hemisphere of $x^2+y^2+z^2=9$. [Answer: $(0,0,3/2)$]

18. Find the centroid of $z = x^2+y^2$ where $z \leq 1$.

19. Let $\mathbf{X}(u,v) = (u\cos(v), u\sin(v), v)$ where $0 \leq u \leq 1$ and $0 \leq v \leq 2\pi$. Let S_1 be the surface parameterized by $\mathbf{X}$. Graph S_1 in Maple. Then evaluate $\iint_{S_1} (x,y,z-2y)\cdot d\mathbf{S}$ if S_1 is given the orientation $\mathbf{n} = \dfrac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}$. [Answer: π^2]

20. Let S_1 be the part of the paraboloid $z = 1 - x^2 - y^2$ which lies above the xy-plane ($z \geq 0$) and orient S_1 upward. Evaluate $\iint_{S_1} (x,y,z)\cdot d\mathbf{S}$. [Answer: $3\pi/2$]

21. Let S_1 be the upper-hemisphere of $x^2+y^2+z^2=9$ oriented upward. Evalute $\iint_{S_1} (-y,x,-1)\cdot d\mathbf{S}$. [Answer: -9π]

22. Let S_1 be the surface of the cylinder bounded by $x^2+y^2=4$, $z=0$, and $z=5$ oriented outward (include the top and the bottom of the cylinder). Evaluate $\iint_{S_1} (x^3,y^3,0)\cdot d\mathbf{S}$. If you know the Divergence Theorem, use it to recompute your answer. [Answer: 120π]

23. Let S_1 be the part of the plane $x+y+z=1$ which lies in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) and oriented **downward**. Evaluate $\iint_{S_1} (xze^y\mathbf{i} - xze^y\mathbf{j} + z\mathbf{k})\cdot d\mathbf{S}$. [Answer: $-1/6$]

24. Let S_1 be the cube with vertices $(\pm1,\pm1,\pm1)$ oriented outward. Evaluate $\iint_{S_1} (x,2y,3z)\cdot d\mathbf{S}$. If you know the divergence theorem, recalculate this integral using the theorem. [Answer: 48]

Stokes' Theorem:

25. Let S be the lower half of the sphere $x^2+y^2+z^2=4$ oriented downward with boundary C. Also, let $\mathbf{F}(x,y,z)=(2y-z, x+y^2-z, 4y-3x)$. Verify Stokes' Theorem by computing both sides of $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{X} = \iint_S \text{curl}(\mathbf{F})\cdot d\mathbf{S}$.

26. Verify Stoke's theorem for $\mathbf{F}(x,y,z) = (y^2, x, z^2)$ and S_1 the part of the circular paraboloid $z = x^2+y^2$ which lies below $z=1$. Give S_1 the upward orientation.

27. Let C be the circle parameterized by $\mathbf{X}(t) = (\cos(t), \sin(t), 0)$ where $0 \le t \le 2\pi$. Evaluate $\displaystyle\int_C (z^2, 2x, -y^3)\cdot d\mathbf{X}$. [Answer: 2π]

28. Let C be the rectangular boundary of the part of the plane $z = y$ which lies above $0 \le x \le 1$ and $0 \le y \le 3$. Give C the counter-clockwise orientation when viewed from above. Evaluate $\displaystyle\int_C (x^2, 4xy^3, y^2x)\cdot d\mathbf{X}$. [Answer: 90]

29. Let S_1 be the part of the graph of $z = e^{-(x^2+y^2)}$ which lies above $x^2+y^2 \le 1$. Orient S_1 upward. Let $\mathbf{F}(x,y,z) = (e^{y+z}-2y)\mathbf{i} + (xe^{y+z}+y)\mathbf{j} + e^{x+y}\mathbf{k}$. Evaluate $\displaystyle\iint_{S_1} \text{curl}(\mathbf{F})\cdot d\mathbf{S}$. Hint: Use Stokes' theorem to replace S_1 with another surface which shares the same boundary. [Answer: 2π]

30. S_1 be the four sides and the top of the cube with vertices $(\pm1, \pm1, \pm1)$ oriented outward. Let $\mathbf{F}(x,y,z) = xyz\mathbf{i} + xy\mathbf{j} + x^2yz\mathbf{k}$. Evaluate $\displaystyle\iint_{S_1} \text{curl}(\mathbf{F})\cdot d\mathbf{S}$ by using Stokes' theorem to exchange S_1 with an "easier" surface whose boundary is the same as S_1. [Answer: 0]

31. Compute the work done by the force field $\mathbf{F}(x,y,z) = (x^x+z^2, y^y+x^2, z^z+y^2)$ when a particle moves under its influence around the edge of the part of the sphere $x^2+y^2+z^2=4$ which lies in the 1^{st} octant and is oriented in a counter-clockwise direction when viewed from above. [Answer: 16]

32. Let C be the curve of intersection of $x+y+z=1$ and $x^2+y^2=9$. Orient C counter-clockwise when viewed from above. Evaluate $\displaystyle\int_C (x^2z, xy^2, z^2)\cdot d\mathbf{X}$. [Answer: $81\pi/2$]

The Divergence Theorem:

33. Let E be the solid bounded above by $z = 9 - x^2 - y^2$ and below by $z = 0$, let S be the boundary of E, and let $\mathbf{F}(x, y, z) = (x, y, z)$. Verify the Divergence Theorem by computing both sides of $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}(\mathbf{F})\, dV$.

34. Let S_1 be the unit sphere $x^2 + y^2 + z^2 = 1$ oriented outward. Evaluate $\iint_{S_1} (z, y, x) \cdot d\mathbf{S}$. [Answer: $4\pi/3$]

35. Verify the divergence theorem when $\mathbf{F}(x, y, z) = (xy, yz, zx)$ and E is the cylindrical region bounded by $x^2 + y^2 = 1$, $z = 0$, and $z = 1$.

36. Let S_1 be the boundary of the upper-half of $x^2 + y^2 + z^2 \leq 1$ (the upper-half of the unit ball). Orient S outward. Evaluate $\iint_{S_1} (x^3, y^3, z^3) \cdot d\mathbf{S}$. [Answer: $6\pi/5$]

37. Let S_1 be the surface of the circular paraboloid bounded by $z = x^2 + y^2$ and $z = 1$. Orient S_1 outward. Evaluate $\iint_{S_1} (x^2, y^2, z^2) \cdot d\mathbf{S}$. [Answer: $2\pi/3$]

38. Let S_1 be the surface of the cylinder bounded by $y^2 + z^2 = 1$, $x = -1$, and $x = 2$ (including the front and back) oriented outward. Evaluate $\iint_{S_1} (3xy^2, xe^z, z^3) \cdot d\mathbf{S}$. [Answer: $9\pi/2$]

39. Let S_1 be the upper-hemisphere of $x^2 + y^2 + z^2 = 1$ oriented upwards (this is just the upper shell do not include the bottom). Also, let $\mathbf{F}(x, y, z) = (z^2 x, y^3/3 + \tan(z), x^2 z + y^2)$. Evaluate $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$. Hint: use the theorem to exchange S_1 with an "easier" surface. [Answer: $13\pi/20$]

40. Let S_1 be the part of the graph of $z = (1 - x^2 - y^2)e^{1-x^2-3y^2}$ which lies above the xy-plane oriented upward. Also, let $\mathbf{F}(x, y, z) = (e^y \cos(z), \sqrt{x^3 + 1}\sin(z), x^2 + y^2 + 3)$. Evaluate $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$ by using the theorem to exchange S_1 with an "easier" surface. [Answer: $7\pi/2$]

41. Prove Green's Formulas: Let f and g be smooth functions and let S be the surface (oriented outward) of a simple solid region E.

$$\iiint_E \nabla f \cdot \nabla g + f\, \text{div}(\nabla g)\, dV = \iint_S f\, \nabla g \cdot d\mathbf{S}$$

$$\iiint_E (f\, \text{div}(\nabla g) - g\, \text{div}(\nabla f))\, dV = \iint_S (f\, \nabla g - g\, \nabla f) \cdot d\mathbf{S}$$

Hint: First, prove the following "product" rule: $\text{div}(f\,\mathbf{F}) = \nabla f \cdot \mathbf{F} + f\, \text{div}(\mathbf{F})$.