Differential Geometry & Line Integrals: Notation & Formulas Review

- Let C be an oriented curve parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ where $a \leq t \leq b$.
- $-C$ denotes the curve C with its orientation (i.e., direction) reversed. We let $C_1 + C_2$ denote curve C_1 and curve C_2 joined together. Such curves don't have to be connected. More generally, $-3C_1 + 5C_2$ is the curves where we travel along C_1 backwards 3 times and then run along C_2 a total of 5 times.
- Define $s(t) = \displaystyle\int_a^t |\mathbf{r}'(u)|\, du$. This is the *arc length* function.

  Note: $s(a) = 0$, $\boxed{s(b) = \displaystyle\int_a^b |\mathbf{r}'(t)|\, dt}$ is the total arc length of C, and by the second part of the fundamental theorem of calculus $\dfrac{ds}{dt} = |\mathbf{r}'(t)|$ so that we define $\boxed{ds = |\mathbf{r}'(t)|\, dt}$ to be the arc length element used in the definition of line integrals with respect to arc length.
- $\boxed{\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}}$ is the *unit Tangent*, $\boxed{\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}}$ is the *unit Normal*, and $\boxed{\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)}$ is the *Binormal*. For every t (where these are defined), they are mutually perpendicular (i.e., $\mathbf{T} \bullet \mathbf{N} = 0$, $\mathbf{N} \bullet \mathbf{B} = 0$, and $\mathbf{B} \bullet \mathbf{T} = 0$) unit vectors (i.e., $|\mathbf{T}| = |\mathbf{N}| = |\mathbf{B}| = 1$). This TNB-frame forms a right handed system – much like a variable **ijk** triple. In particular, $\mathbf{T} \times \mathbf{N} = \mathbf{B}$, $\mathbf{N} \times \mathbf{B} = \mathbf{T}$, and $\mathbf{B} \times \mathbf{T} = \mathbf{N}$. We can think of $\mathbf{T}$ as pointing forward, $\mathbf{N}$ pointing left, and $\mathbf{B}$ pointing up. Just like decomposing a vector into $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$-components, every vector can be decomposed into $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$-components.
- Curvature is given by $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$. The first formula shows that κ depends only on the shape of the curve (not the particular parameterization). The second formula is easy to use when the TNB-frame has already been computed. Generally, the third formula is easiest to apply if we are computing curvature from scratch.

  Curvature measures how bent a curve is. If C is a circle of radius R, we have $\boxed{\kappa = \dfrac{1}{R}}$ (larger circles are less bent). We also have $\boxed{\kappa(t) = 0 \text{ for all } t \iff C \text{ is a line or a line segment}}$.

  Note: Even when $\mathbf{r}(t)$ is many times differentiable, its TNB-frame may not exist (at least at certain points). Specifically, if $\mathbf{T}'(t_0) = \mathbf{0}$, we cannot normalize $\mathbf{T}'(t_0) = \mathbf{0}$ and so the unit normal and binormal will not exist for such a $t = t_0$. In fact, keeping in mind $\kappa = |\mathbf{T}'|/|\mathbf{r}'|$, we have that a sufficiently differential curve will have a well-defined TNB-frame at $t = t_0$ if and only if $\kappa(t_0) \neq 0$.
- For a particular $t = t_0$, the plane which is parallel to $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$ (and thus perpendicular to $\mathbf{B}(t_0)$) and through the point $\mathbf{r}(t_0)$ is called the *osculating plane* at $\mathbf{r}(t_0)$. Such a plane has vector formula, $\mathbf{B}(t_0) \bullet (\langle x, y, z \rangle - \mathbf{r}(t_0)) = 0$. [osculating plane = kissing plane]

  We can paramterize the line tangent to C at $\mathbf{r}(t_0)$ by $\vec{\ell}(t) = \mathbf{r}(t_0) + \mathbf{r}'(t_0)t$ or $\vec{\ell}(t) = \mathbf{r}(t_0) + \mathbf{T}(t_0)t$. The circular version of the tangent line is called a osculating circle or circle of curvature. The *osculating circle* at $\mathbf{r}(t_0)$ is the circle lying in the osculating plane at $\mathbf{r}(t_0)$ whose curvature matches the curvature of C at that point and whose center lies in the unit normal direction from $\mathbf{r}(t_0)$. Specifically, this circle can be parameterized by...

$$\mathbf{c}(t) = \underbrace{\mathbf{r}(t_0) + \frac{1}{\kappa(t_0)}\mathbf{N}(t_0)}_{\text{the circle's center}} + \frac{1}{\kappa(t_0)}\cos(t)\mathbf{T}(t_0) + \frac{1}{\kappa(t_0)}\sin(t)\mathbf{N}(t_0)$$

- If we think of $\mathbf{r}(t)$ as the position of a particle, then $\mathbf{v}(t) = \mathbf{r}'(t)$ is its velocity, $|\mathbf{r}'(t)|$ is its speed, and $\mathbf{a}(t) = \mathbf{r}''(t)$ is its acceleration. Decomposing acceleration into its TNB-frames components we have $\boxed{\mathbf{r}''(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t) + 0\mathbf{B}(t)}$ where $a_T(t)$ and $a_N(t)$ are the *tangential and normal components of acceleration*. Notice that the binormal component of acceleration is always identically 0 (this has important physical consequences).

  Note: We have direct formula for these components. $a_T(t) = \dfrac{\mathbf{r}'(t) \bullet \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$ and $a_N(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$. If you have already computed curvature using the cross product formula, you've nearly computed these components as well.

- Using $\kappa = |\mathbf{T}'|/|\mathbf{r}'|$ and $\mathbf{N} = \mathbf{T}'/|\mathbf{T}'|$, is immediately follows that $\mathbf{T}'(t) = \kappa(t)|\mathbf{r}'(t)|\mathbf{N}(t)$. It turns out that $\mathbf{B}'(t)$ is also parallel to $\mathbf{N}(t)$. We can define a function $\tau(t)$, called the *torsion* of C, by requiring $\boxed{\mathbf{B}'(t) = -\tau(t)|\mathbf{r}'(t)|\mathbf{N}(t)}$. It can also be shown that $\mathbf{N}'(t) = -\kappa(t)|\mathbf{r}'(t)|\mathbf{T}(t) + \tau(t)|\mathbf{r}'(t)|\mathbf{B}(t)$. These formulas for the derivatives of the $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are called the Frenet-Serret formulas.

  Torsion is a measurement of how the binormal changes. Since the binormal determines how the osculating plane is tilted, if $\mathbf{B}'(t) = \mathbf{0}$, then $\mathbf{B}(t)$ is constant and thus the osculating plane is constant as well. Since points on our curve near $\mathbf{r}(t_0)$ nearly lie in the osculating plane at $\mathbf{r}(t_0)$, we can only have a constant osculating plane when our curve is planar. Therefore, $\boxed{\tau(t) = 0 \text{ for all } t \iff \mathbf{B}'(t) = \mathbf{0} \iff \mathbf{B}(t) \text{ is constant} \iff C \text{ is a planar curve}}$.

  Much like curvature, there is a relatively simple formula for torsion: $\tau(t) = \dfrac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \bullet \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$.

- Line integrals with respect to arc length (i.e., of scalar valued functions) compute net area of a sheet under a surface (or hyper-surface): $\boxed{\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt}$. Specifically, we let $ds = |\mathbf{r}'(t)|\,dt = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$, $x = x(t)$, $y = y(t)$, and $z = z(t)$.
  Note: The value of $\int_C f(x,y,z)\,ds$ is independent of the choice of paramterization for C. In fact, it does not even depend on the orientation of C: $\int_{-C} f(x,y,z)\,ds = \int_C f(x,y,z)\,ds$.

  Just as $\int_a^b 1\,dx = b - a$ is the length of the interval $I = [a,b]$, we have $\boxed{\displaystyle\int_C 1\,ds = \text{Arc Length of } C}$.

Center of mass

Suppose we have a wire bent in the shape of the curve C and suppose this wire has density $\delta(x,y,z)$ at each point (x,y,z) along the curve. Then if we focus on a little segment of the wire where the density is roughly constant, the mass of the segment of wire will be approximately $\delta(x_0,y_0,z_0)\Delta s$ where Δs is the length of this piece of the wire. So if we add up $\Sigma\delta\Delta s$ we should get the total mass of the wire (approximately anyway). Translating to the world of integrals we have...

- The total *mass* of the wire is $m = \displaystyle\int_C \delta(x,y,z)\,ds$
- Let $M_{yz} = M_{x=0} = \displaystyle\int_C x\,\delta(x,y,z)\,ds$. We call M_{yz} the *moment about the yz-plane*. This is a weighted sum of x-coordinates. Likewise, $M_{xz} = M_{y=0} = \displaystyle\int_C y\,\delta(x,y,z)\,ds$ and $M_{xy} = M_{x=0} = \displaystyle\int_C z\,\delta(x,y,z)\,ds$ are *moments about the xz- and xy-planes* respectively. These compute weighted sums of y and z coordinates.
- Finally, $(\bar{x},\bar{y},\bar{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right)$ is the *center of mass* of C with density function δ. The coordinates of the center of mass are weighted averages of x, y, and z coordinates on the curve.
- If $\delta(x,y,z) = c =$ constant $\neq 0$, then δ will cancel out of the computation of the center of mass (in such a case we can just take $\delta = 1$ in the formulas). In the case of a constant density function, we call the center of mass $(\bar{x},\bar{y},\bar{z})$ the *centroid* of C. This is a geometric center of our curve.

Example: Let C be the helix parameterized by $\mathbf{r}(t) = \langle \cos(t), \sin(t), \sqrt{3}t\rangle$ for $0 \leq t \leq 2\pi$. We will find the centroid of C (i.e., let $\delta = 1$).

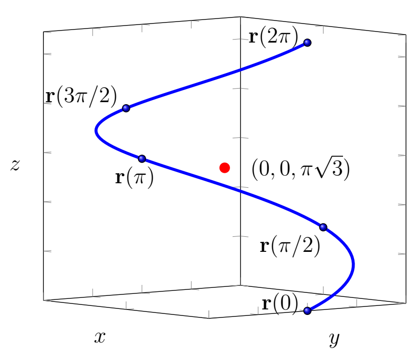
First, $\mathbf{r}'(t) = \langle -\sin(t), \cos(t), \sqrt{3}\rangle$ so $|\mathbf{r}'(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (\sqrt{3})^2} = \sqrt{4} = 2$. Thus $ds = 2\,dt$. The mass (= arc length) of C is $m = \displaystyle\int_C 1\,ds = \int_0^{2\pi} 2\,dt = 4\pi$

$$\bar{x} = \frac{1}{m}\int_C x\,ds = \frac{1}{4\pi}\int_0^{2\pi} \cos(t)2\,dt = 0 \qquad \bar{y} = \frac{1}{m}\int_C y\,ds = \frac{1}{4\pi}\int_0^{2\pi} \sin(t)2\,dt = 0$$

$$\bar{z} = \frac{1}{m}\int_C z\,ds = \frac{1}{4\pi}\int_0^{2\pi} \sqrt{3}t2\,dt = \frac{\sqrt{3}}{4\pi}\int_0^{2\pi} 2t\,dt = \frac{\sqrt{3}}{4\pi}t^2\Big|_0^{2\pi} = \frac{\sqrt{3}}{4\pi}(2\pi)^2 - 0 = \pi\sqrt{3} \approx 5.4414$$

Notice the helix's z-coordinates range from $z = 0$ to $z = \sqrt{3}\cdot 2\pi$ so $\bar{z}$ is exactly half way between 0 and $\sqrt{3}\cdot 2\pi$ (looking at a picture of this helix should convince you that this is the right answer).

Answer: $(\bar{x},\bar{y},\bar{z}) = (0,0,\pi\sqrt{3})$



Supplemental Problems:

1. $\displaystyle\int_C 3x^2yz\,ds$ where C is the curve parameterized by $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3 \right\rangle$ and $0 \le t \le 1$.
2. $\displaystyle\int_C xy^4\,ds$ where C is the right half of the circle centered at the origin of radius 2.
3. $\displaystyle\int_C x^2z\,ds$ where C is the line segment from $(0,6,-1)$ to $(4,1,5)$.
4. $\displaystyle\int_C \frac{x}{1+y^2}\,ds$ where C is the line segment $x = 1+2t$, $y = t$ where $0 \le t \le 1$.
5. $\displaystyle\int_C \frac{e^{-z}}{x^2+y^2}\,ds$ where C is the helix $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t), t\rangle$ and $0 \le t \le 2\pi$.
6. Find the centroid of the helix C param. by $\mathbf{r}(t) = \langle 4\sin(t), 3t, 4\cos(t)\rangle$ where $-2\pi \le t \le 4\pi$.
7. Find the centroid of the helix C param. by $\mathbf{r}(t) = \langle 2\sin(t), 2\cos(t), 3t\rangle$ where $0 \le t \le \pi$.
8. Find the centroid of the right half of the circle $x^2 + y^2 = 9$.
9. Find the centroid of the part of the circle $x^2 + y^2 = 25$ in the first quadrant.
10. Find the centroid of the curve C: the upper-half of the unit circle plus the x-axis from -1 to 1.
    Hint: Use geometry and symmetry to compute 2 of the 3 line integrals.

Answers:

1. $ds = 2t^2 + 1\,dt$ $\quad \displaystyle\int_0^1 (4t^9 + 2t^7)\,dt = \boxed{\frac{13}{20}}$
2. $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle$, $-\pi/2 \le t \le \pi/2$, $ds = 2\,dt$ $\quad \displaystyle\int_{-\pi/2}^{\pi/2} 64\sin^4(t)\cos(t)\,dt = \boxed{\frac{128}{5}}$
3. $\mathbf{r}(t) = \langle 0,6,-1\rangle + \langle 4,-5,6\rangle t$, $0 \le t \le 1$, $ds = \sqrt{77}\,dt$ $\quad \displaystyle\int_0^1 16t^2(6t-1)\sqrt{77}\,dt = \boxed{\frac{56\sqrt{77}}{3}}$
4. $ds = \sqrt{5}\,dt$ $\quad \displaystyle\int_0^1 \frac{2t+1}{t^2+1}\sqrt{5}\,dt = \boxed{\sqrt{5}\ln(2) + \frac{\sqrt{5}}{4}\pi}$
5. $ds = \sqrt{5}\,dt$ $\quad \displaystyle\int_0^{2\pi} \frac{\sqrt{5}}{4}e^{-t}\,dt = \boxed{\frac{\sqrt{5}}{4}(1 - e^{-2\pi})}$
6. $ds = 5\,dt$, $m = 30\pi$ $\quad \boxed{(\bar{x}, \bar{y}, \bar{z}) = (0, 3\pi, 0)}$
7. $ds = \sqrt{13}\,dt$, $m = \sqrt{13}\pi$ $\quad \boxed{(\bar{x}, \bar{y}, \bar{z}) = (4/\pi, 0, 3\pi/2)}$
8. $ds = 3\,dt$, $m = 3\pi$ $\quad \boxed{(\bar{x}, \bar{y}) = (6/\pi, 0)}$
9. $ds = 5\,dt$, $m = 5\pi/2$ $\quad \boxed{(\bar{x}, \bar{y}) = (10/\pi, 10/\pi)}$
10. $m = \pi + 2$ (half-circle plus line segment), symmetry: $\bar{x} = 0$, $M_x = \int_0^\pi \sin(t)\,dt + \int_{-1}^1 0\,dt = 2$
    $\boxed{(\bar{x}, \bar{y}) = (0, 2/(2+\pi))}$