Math 2130

Riemann Sums

Supplemental Homework

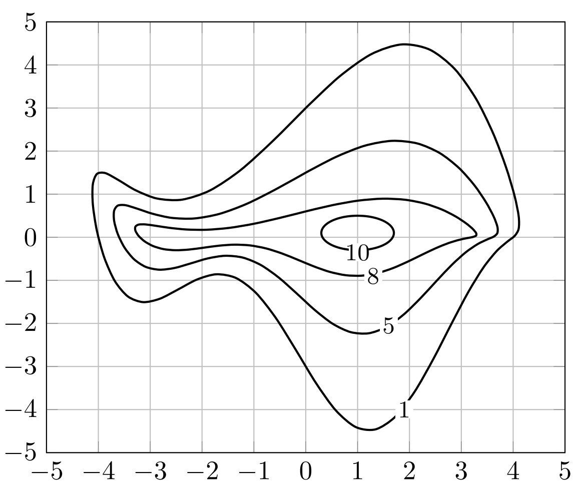
Approximate $I = \iint\limits_R f(x,y)\,dA$ for the specified rectangle R and function f with the given evaluation method using a grid of $m \times n$ rectangles (m across and n up). First, figure out Δx and Δy. Then draw a grid and specify the sample points (where we evaluate $f(x,y)$), then write down the approximation. Don't worry about actually crunching numbers or simplifying.

1. $R = [0,6] \times [0,10]$ and $f(x,y) = x^2 y$ using midpoint rule and 3×2 rectangles.
2. $R = [-1,11] \times [-4,4]$ and $f(x,y) = \ln(x + y^2 + 3)$ using midpoint rule and 2×2 rectangles.
3. $R = [-2,6] \times [1,7]$ and $f(x,y) = \sqrt{x^2 + y}$ using midpoint rule and 2×3 rectangles.
4. $R = [-4,2] \times [-3,3]$ and $f(x,y) = y\sin(x)$ using midpoint rule and 3×3 rectangles.

5–8. Redo problems 1.–4. but use a lower-lefthand rule instead of midpoint rule. [By "lower-lefthand rule" I mean that you should evaluate using points built from left endpoints in both the x- and y-coordinate partitions.] You could also practice using other evaluation rules, but hopefully this gets the point across.

9. Consider the approximations done in problems 1. and 5. It is easy enough to directly compute $\iint\limits_{[0,6]\times[0,10]} x^2 y\,dA$ and find that the exact answer is 3600 (do this). If we use a grid of $n \times n$ rectangles and midpoint rule, how large does n have to get before our approximation is within 5 of the actual answer? [*Note:* You probably should involve some technology here. Doing this "by hand" would be incredibly tedious.]

10.



The graph on left displays the level curves of a pile of gravel. Suppose the measurements are in feet. Estimate the volume of this pile using a Riemann sum.

Note: Two people working on this problem should not expect to arrive at the exact same estimate. First, we need to make a judgment call as what size grid to use. Next, when sampling heights of the pile, we need to estimate given the partial information provided by the graph. For example, the point $(x,y) = (1,0)$ is probably the top of the pile, so a height of 10.25 feet might be reasonable. But this is just a "guesstimate".

11. You have a $18' \times 10'$ rectangular swimming pool in your backyard and want to estimate how much water it holds. You grid off the pool into 3×2 subrectangles. In each subrectangular section of your pool you use a dipstick to determine how deep it is. Here are your results: $2'$, $4.5'$, $5'$, $3'$, $5'$, and $5.5'$.

    Approximately how much water does your pool hold?

Answers: [Except, I'm not going to sketch grids for you.]

1. $\Delta x = 2$, $\Delta y = 5$, $I \approx 2 \cdot 5 \cdot \left(1^2(2.5) + 3^2(2.5) + 5^2(2.5) + 1^2(7.5) + 3^2(7.5) + 5^2(7.5)\right) = 3500$

2. $\Delta x = 6$, $\Delta y = 4$, $I \approx 6 \cdot 4 \cdot \left(\ln(2 + (-2)^2 + 3) + \ln(8 + (-2)^2 + 3) + \ln(2 + 2^2 + 3) + \ln(8 + 2^2 + 3)\right) \approx 235.453$

3. $\Delta x = 4$, $\Delta y = 2$, $I \approx 4 \cdot 2 \cdot \left(\sqrt{0^2+2} + \sqrt{4^2+2} + \sqrt{0^2+4} + \sqrt{4^2+4} + \sqrt{0^2+6} + \sqrt{4^2+6}\right) \approx 154.151$

4. $\Delta x = 2$, $\Delta y = 2$, $I \approx 2 \cdot 2 \cdot \Big((-2)\sin(-3) + (-2)\sin(-1) + (-2)\sin(1) + (0)\sin(-3) + (0)\sin(-1)$
$+ (0)\sin(1) + (2)\sin(-3) + (2)\sin(-1) + (2)\sin(1)\Big) = 0$

5. $\Delta x = 2$, $\Delta y = 5$, $I \approx 2 \cdot 5 \cdot \left(0^2(0) + 2^2(0) + 4^2(0) + 0^2(5) + 2^2(5) + 4^2(5)\right) = 1000$

6. $\Delta x = 6$, $\Delta y = 4$, $I \approx 6 \cdot 4 \cdot \left(\ln((-1) + (-4)^2 + 3) + \ln(5 + (-4)^2 + 3) + \ln((-1) + 0^2 + 3) + \ln(5 + 0^2 + 3)\right) \approx 212.184$

7. $\Delta x = 4$, $\Delta y = 2$, $I \approx 4 \cdot 2 \cdot \left(\sqrt{(-2)^2+1} + \sqrt{2^2+1} + \sqrt{(-2)^2+3} + \sqrt{2^2+3} + \sqrt{(-2)^2+5} + \sqrt{2^2+5}\right) \approx 126.109$

8. $\Delta x = 2$, $\Delta y = 2$, $I \approx 2 \cdot 2 \cdot \Big((-3)\sin(-4) + (-3)\sin(-2) + (-3)\sin(0) + (-1)\sin(-4) + (-1)\sin(-2)$
$+ (-1)\sin(0) + (1)\sin(-4) + (1)\sin(-2) + (1)\sin(0)\Big) = 1.830$

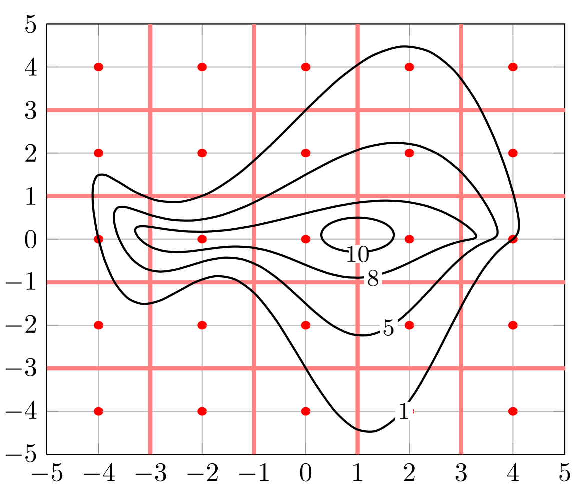
9. For the exact answer: $\displaystyle\int_0^{10}\int_0^6 x^2 y\,dx\,dy = \int_0^{10} \frac{1}{3}x^3 y\Big|_0^6 = \int_0^{10}\left(\frac{6^3}{3}y - 0\right)dy = \int_0^{10} 72y\,dy = 36y^2\Big|_0^{10} = 3600.$

   Using Maple I found that using a grid of 13×13 rectangles gave $I \approx 3594.675$ but 14×14 rectangles gave 3595.408. So we need $n = 14$ to get within 5. Sample Code:

```
for N from 1 to 20 do
  dx := (6-0)/N:  dy := (10-0)/N:
  S := evalf(sum(sum( ((i-1)*dx+dx/2)^2 * ((j-1)*dy+dy/2), i=1..N), j=1..N)*dx*dy);
  print(N,S);
end do:
```

   Notice that in the above code `(i-1)*dx` would be the left endpoint of the i-th subinterval in the partition of the x-coordinate, so `(i-1)*dx+dx/2` would give the i-th midpoint.

10. This is a sample solution. Again, a lot of choices must be made, so no two solutions are likely to match.



   I will start with a 5×5 grid. Thus $\Delta A = \Delta x \cdot \Delta y = 2 \cdot 2 = 4$. I will also use midpoint rule. This still leaves some variability since we don't have a formula to evaluate. We must estimate $f(x, y)$'s value at our sample points as best we can from the given graph. The calculation below runs through the points left to right, top to bottom.

   Volume $\approx \Delta A \cdot \Big(f(-4,-4) + f(-2,-4) + f(0,-4) + f(2,-4)$
   $+ f(4,-4) + f(-4,-2) + \cdots + f(4,4)\Big)$

   $$\approx 4(\; 0 + 0 + 0.25 + 2 + 0 + \\ 0.5 + 0 + 3.5 + 5.5 + 0.75 + \\ 1 + 9 + 9 + 9 + 1 + \\ 0 + 0 + 3 + 4.5 + 0 + \\ 0 + 0 + 0.5 + 1 + 0 \;) = 202 \text{ cubic feet}$$

   We have approximately 202 cubic feet of gravel.

11. The pool is 18 by 10 feet. If we use a 3×2 grid, we should have $\Delta x = 18/3 = 6$ and $\Delta y = 10/2 = 5$ so that $\Delta A = 6 \cdot 5 = 30$ square feet per subrectangle. Thus the volume is about $30 \cdot (2 + 4.5 + 5 + 3 + 5 + 5.5) = 30 \cdot 25 = 750$ cubic feet. Thus your pool holds approximately 5,610 gallons of water.

    Note: If you try to draw a diagram of this pool with depths noted, it is unclear which depth measurement go where. But no matter, the order of these measurements has no impact on the answer.