Math 2130-101 # Quiz #7 – Answers **Nov. 30th, 2022**

#1. Let $\mathbf{F}(x,y,z) = \langle x^3+yz, x\sin(y^2z), 5\rangle$.

(a) Compute the divergence of $\mathbf{F}$:

$$\nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}\left[x^3+yz\right] + \frac{\partial}{\partial y}\left[x\sin(y^2z)\right] + \frac{\partial}{\partial z}[5] = \boxed{3x^2+2xyz\cos(y^2z)} + 0$$

(b) Compute the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3+yz & x\sin(y^2z) & 5 \end{vmatrix} = \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ x\sin(y^2z) & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ x^3+yz & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ x^3+yz & x\sin(y^2z) \end{vmatrix}\mathbf{k}$$

$$= \left\langle \frac{\partial}{\partial y}[5] - \frac{\partial}{\partial z}\left[x\sin(y^2z)\right], -\left(\frac{\partial}{\partial x}[5] - \frac{\partial}{\partial z}\left[x^3+yz\right]\right), \frac{\partial}{\partial x}\left[x\sin(y^2z)\right] - \frac{\partial}{\partial y}\left[x^3+yz\right]\right\rangle$$

$$= \boxed{\left\langle -xy^2\cos(y^2z), y, \sin(y^2z)-z\right\rangle}$$

Unnecessary note: $\nabla \times \mathbf{F} \neq \mathbf{0}$ so $\mathbf{F}$ is not conservative.

#2. Set up the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = \langle y^2+z^2, x, yz\rangle$ and C is parameterized by $\mathbf{r}(t) = \langle 3t, 2\cos(t), 2\sin(t)\rangle$ for $-\pi \leq t \leq 5\pi$.

We already have a parameterization for C, so our next step to compute its derivative: $\mathbf{r}'(t) = \langle 3, -2\sin(t), 2\cos(t)\rangle$. Then, we plug everything in. Keep in mind that our parameterization (i.e., $\mathbf{r}(t)$) tells us that $x = 3t$, $y = 2\cos(t)$, and $z = 2\sin(t)$ and so $y^2+z^2 = 4$. Therefore,

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{-\pi}^{5\pi} \langle 4, 3t, 2\cos(t)\cdot 2\sin(t)\rangle \bullet \langle 3, -2\sin(t), 2\cos(t)\rangle\, dt = \boxed{\int_{-\pi}^{5\pi} 12 - 6t\sin(t) + 8\cos^2(t)\sin(t)\, dt}$$

Note: We were not asked to evaluate this integral, but it isn't too difficult. The $-6t\sin(t)$ term requires integration by parts (which we have almost completely avoided in this class) and the final term can be tackled with a simple u-substitution: $u = \cos(t)$ and $du = -\sin(t)\,dt$. For those curious, the value of this integral is 36π.

#3. Let $\mathbf{F} = \langle yz+1, z^3+xz, 3yz^2+xy+e^{-z}\rangle$. *Note:* $\mathbf{F}$ is conservative: $\nabla \times \mathbf{F} = \mathbf{0}$ (I've checked for you).

(a) Find a potential function for $\mathbf{F}$. If $\mathbf{F} = \nabla f$, we must have $f_x = yz+1$, $f_y = z^3+xz$, and $f_z = 3yz^2+xy+e^{-z}$. Therefore, $f = \int(yz+1)\,dx = xyz+x+C_1(y,z)$ and $f = \int(z^3+xz)\,dy = yz^3+xyz+C_2(x,z)$ and $f = \int(3yz^2+xy+e^{-z})\,dz = yz^3+xyz-e^{-z}+C_3(x,y)$.

We now merge these results to reconstruct f. *Note:* Any term involving 3 variables must appear 3 times (likewise 2 variable terms appear twice and 1 variable terms appear only once). Therefore, $\boxed{f(x,y,z) = xyz+x+yz^3-e^{-z}}$ [Adding an arbitrary constant "$+C$" will yield all possible potential functions.]

(b) Use the fundamental theorem of line integrals to evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ if C is a curve from $(1,-1,0)$ to $(0,1,0)$.

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = f(\text{end of } C) - f(\text{start of } C) = f(0,1,0) - f(1,-1,0) = (0+0+0-e^0) - (0+1+0-e^0) = \boxed{-1}$$

Note: The fundamental theorem of line integrals was *not* an option for evaluating the integral in problem #2 since that vector field $\mathbf{F}$ was not conservative (so it did not have any associated potential functions). It is also worth noting that the integral in this problem could be computed directly. We could parameterize *any* curve from $(1,-1,0)$ to $(0,1,0)$ (for example, a line segment) and compute the value that way (since conservative vector fields have path independent line integrals). However, this would be more work than just applying our fundamental theorem.

Grading Notes: #1: 1.5 pts part (a) and 2 pts part (b) #2: 3 pts #3: 2.5 pts part (a) and 1 pt part (b).

Math 2130-102 **Quiz #7 – Answers** **Nov. 30$^{\text{th}}$, 2022**

#1. Let $\mathbf{F}(x,y,z) = \langle x^5y^3z + y^2, yz^3, \sin(yz^2)\rangle$.

(a) Compute the divergence of $\mathbf{F}$:

$$\nabla \bullet \mathbf{F} = \frac{\partial}{\partial x}\left[x^5y^3z+y^2\right] + \frac{\partial}{\partial y}\left[yz^3\right] + \frac{\partial}{\partial z}\left[\sin(yz^2)\right] = \boxed{5x^4y^3z + z^3 + 2yz\cos(yz^2)}$$

(b) Compute the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^5y^3z+y^2 & yz^3 & \sin(yz^2) \end{vmatrix} = \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ yz^3 & \sin(yz^2) \end{vmatrix}\mathbf{i} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ x^5y^3z+y^2 & \sin(yz^2) \end{vmatrix}\mathbf{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ x^5y^3z+y^2 & yz^3 \end{vmatrix}\mathbf{k}$$

$$= \left\langle \frac{\partial}{\partial y}\left[\sin(yz^2)\right] - \frac{\partial}{\partial z}\left[yz^3\right], -\left(\frac{\partial}{\partial x}\left[\sin(yz^2)\right] - \frac{\partial}{\partial z}\left[x^5y^3z+y^2\right]\right), \frac{\partial}{\partial x}\left[yz^3\right] - \frac{\partial}{\partial y}\left[x^5y^3z+y^2\right]\right\rangle$$

$$= \boxed{\left\langle z^2\cos(yz^2) - 3yz^2, x^5y^3, -3x^5y^2z - 2y\right\rangle}$$

Unnecessary note: $\nabla \times \mathbf{F} \neq \mathbf{0}$ so $\mathbf{F}$ is not conservative.

#2. Set up the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = \langle x^2+z^2, yz, 5\rangle$ and C is parameterized by $\mathbf{r}(t) = \langle 3\cos(t), t, 3\sin(t)\rangle$ for $-3\pi \leq t \leq 4\pi$.

We already have a parameterization for C, so our next step to compute its derivative: $\mathbf{r}'(t) = \langle -3\sin(t), 1, 3\cos(t)\rangle$. Then, we plug everything in. Keep in mind that our parameterization (i.e., $\mathbf{r}(t)$) tells us that $x = 3\cos(t)$, $y = t$, and $z = 3\sin(t)$ and so $x^2 + z^2 = 9$. Therefore,

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{-3\pi}^{4\pi} \langle 9, t\cdot 3\sin(t), 5\rangle \bullet \langle -3\sin(t), 1, 3\cos(t)\rangle\,dt = \boxed{\int_{-3\pi}^{4\pi} -27\sin(t) + 3t\sin(t) + 15\cos(t)\,dt}$$

Note: We were not asked to evaluate this integral, but it isn't too difficult. The $3t\sin(t)$ term requires integration by parts (which we have almost completely avoided in this class). For those curious, the value of this integral is $54 - 3\pi$.

#3. Let $\mathbf{F} = \langle yz\cos(x) + 1, z\sin(x) + z, y\sin(x) + y + e^{-z}\rangle$. *Note:* $\mathbf{F}$ is conservative: $\nabla \times \mathbf{F} = \mathbf{0}$ (I've checked for you).

(a) Find a potential function for $\mathbf{F}$. If $\mathbf{F} = \nabla f$, we must have $f_x = yz\cos(x)+1$, $f_y = z\sin(x)+z$, and $f_z = y\sin(x)+y+e^{-z}$. Therefore, $f = \int (yz\cos(x)+1)\,dx = yz\sin(x) + x + C_1(y,z)$ and $f = \int (z\sin(x)+z)\,dy = yz\sin(x) + yz + C_2(x,z)$ and $f = \int (y\sin(x)+y+e^{-z})\,dz = yz\sin(x) + yz - e^{-z} + C_3(x,y)$.

We now merge these results to reconstruct f. *Note:* Any term involving 3 variables must appear 3 times (likewise 2 variable terms appear twice and 1 variable terms appear only once). Therefore, $\boxed{f(x,y,z) = yz\sin(x) + x + yz - e^{-z}}$
[Adding an arbitrary constant "$+C$" will yield all possible potential functions.]

(b) Use the fundamental theorem of line integrals to evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ if C is a curve from $(0,1,0)$ to $(2\pi,1,0)$.

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla f \bullet d\mathbf{r} = f(\text{end of } C) - f(\text{start of } C) = f(2\pi,1,0) - f(0,1,0) = (0 + 2\pi + 0 - e^0) - (0 + 0 + 0 - e^0) = \boxed{2\pi}$$

Note: The fundamental theorem of line integrals was *not* an option for evaluating the integral in problem #2 since that vector field $\mathbf{F}$ was not conservative (so it did not have any associated potential functions). It is also worth noting that the integral in this problem could be computed directly. We could parameterize *any* curve from $(1,-1,0)$ to $(0,1,0)$ (for example, a line segment) and compute the value that way (since conservative vector fields have path independent line integrals). However, this would be more work than just applying our fundamental theorem.

Grading Notes: #1: 1.5 pts part (a) and 2 pts part (b)   #2: 3 pts   #3: 2.5 pts part (a) and 1 pt part (b).