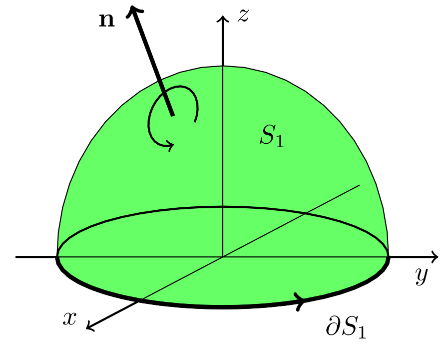
Math 2130

Verifying Stokes

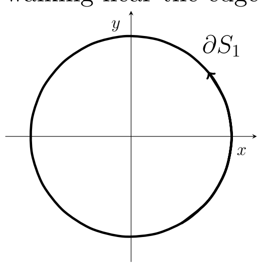
Example

Consider the upper hemisphere S_1: $x^2+y^2+z^2=9$, $z\geq 0$ oriented upward. Also, let $\mathbf{F}(x,y,z)=\langle z,x,x^2+y^2+z^2\rangle$.

We want to show:
$$\underbrace{\iint_{S_1}(\nabla\times\mathbf{F})\bullet\mathbf{n}\,d\sigma}_{\textbf{II.}} \;=\; \underbrace{\int_{\partial S_1}\mathbf{F}\bullet d\mathbf{r}}_{\textbf{I.}}$$



I. The boundary of S_1 (i.e., ∂S_1) is the circle $x^2+y^2=9$ and $z=0$ (i.e., the intersection of the hemisphere with the xy-plane). Considering the orientation for our surface, the orientation of this circle is counter-clockwise (consider someone walking near the edge on top of S_1 trying to keep the surface to their left).



We can parameterize the circle as follows:
∂S_1: $\mathbf{r}(t)=\langle 3\cos(t),3\sin(t),0\rangle$ where $0\leq t\leq 2\pi$. This is properly oriented since our standard parameterization goes counter-clockwise (∂S_1 should be counter-clockwise oriented).

Thus $\mathbf{r}'(t)=\langle -3\sin(t),3\cos(t),0\rangle$ and so $\displaystyle\int_{\partial S_1}\mathbf{F}\bullet d\mathbf{r}=\int_0^{2\pi}\langle 0,3\cos(t),9\rangle\bullet\langle -3\sin(t),3\cos(t),0\rangle\,dt$ since (from our parameterization $\mathbf{r}(t)$) we have $z(t)=0$, $x(t)=3\cos(t)$, and $x(t)^2+y(t)^2+z(t)^2=9+0=9$.

Simplifying, we find that $\displaystyle\int_{\partial S_1}\mathbf{F}\bullet d\mathbf{r}=\int_0^{2\pi}9\cos^2(t)\,dt=\int_0^{2\pi}\frac{9}{2}\Big(1+\cos(2t)\Big)\,dt=\frac{9}{2}t+\frac{9}{4}\sin(2t)\bigg|_0^{2\pi}=\boxed{9\pi}$

II. First, we will compute the curl of our vector field: $\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z & x & x^2+y^2+z^2\end{vmatrix}=\langle 2y-0,-(2x-1),1-0\rangle$.

Thus $\nabla\times\mathbf{F}=\langle 2y,1-2x,1\rangle$.

Next, using spherical coordinates, see that our hemisphere is $\rho^2=9$ (i.e., $\rho=3$) with $0\leq\theta\leq 2\pi$ and $0\leq\varphi\leq\pi/2$ (since $z\geq 0$). Thus we get the parameterization for S_1: $\mathbf{r}(\varphi,\theta)=\langle 3\cos(\theta)\sin(\varphi),3\sin(\theta)\sin(\varphi),3\cos(\varphi)\rangle$. We then need to find the "derivative" of our parameterization:

$$\mathbf{r}_\varphi\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i}\;\mathbf{j}\;\mathbf{k}\\ \mathbf{r}_\varphi\\ \mathbf{r}_\theta\end{vmatrix}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3\cos(\theta)\cos(\varphi) & 3\sin(\theta)\cos(\varphi) & -3\sin(\varphi)\\ -3\sin(\theta)\sin(\varphi) & 3\cos(\theta)\sin(\varphi) & 0\end{vmatrix}=\langle 9\cos(\theta)\sin^2(\varphi),9\sin(\theta)\sin^2(\varphi),9\sin(\varphi)\cos(\varphi)\rangle$$

Note that $\sin(\varphi)$ and $\cos(\varphi)$ are non-negative when $0\leq\varphi\leq\pi/2$, so the $\mathbf{k}$-component of $\mathbf{r}_\varphi\times\mathbf{r}_\theta$ is non-negative. This means that our "derivative" matches the required upward orientation. Also, recall that $\mathbf{n}\,d\sigma=\pm\frac{\mathbf{r}_\varphi\times\mathbf{r}_\theta}{|\mathbf{r}_\varphi\times\mathbf{r}_\theta|}\,|\mathbf{r}_\varphi\times\mathbf{r}_\theta|\,dA=\pm\mathbf{r}_\varphi\times\mathbf{r}_\theta\,dA$ (we keep the $+$). Plugging in our parameterization in we get $(\nabla\times\mathbf{F})(\mathbf{r}(\varphi,\theta))=\langle 2\cdot 3\sin(\theta)\sin(\varphi),1-2\cdot 3\cos(\theta)\sin(\varphi),1\rangle$ since $x(\varphi,\theta)=3\cos(\theta)\sin(\varphi)$ and $y(\varphi,\theta)=3\sin(\theta)\sin(\varphi)$. Putting this altogether we get:

$$\iint_{S_1}(\nabla\times\mathbf{F})\bullet\mathbf{n}\,d\sigma=\int_0^{2\pi}\int_0^{\pi/2}\langle 6\sin(\theta)\sin(\varphi),1-6\cos(\theta)\sin(\varphi),1\rangle\bullet\langle 9\cos(\theta)\sin^2(\varphi),9\sin(\theta)\sin^2(\varphi),9\sin(\varphi)\cos(\varphi)\rangle\,d\varphi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\Big(54\sin(\theta)\cos(\theta)\sin^3(\varphi)+9\sin(\theta)\sin^2(\varphi)-54\sin(\theta)\cos(\theta)\sin^3(\varphi)+9\sin(\varphi)\cos(\varphi)\Big)\,d\varphi\,d\theta$$

$$=\int_0^{\pi/2}\int_0^{2\pi}\Big(9\sin(\theta)\sin^2(\varphi)+9\sin(\varphi)\cos(\varphi)\Big)\,d\theta\,d\varphi=\int_0^{\pi/2}\int_0^{2\pi}\Big(0+9\sin(\varphi)\cos(\varphi)\Big)\,d\theta\,d\varphi$$

$$=\int_0^{\pi/2}18\pi\sin(\varphi)\cos(\varphi)\,d\varphi=9\pi\sin^2(\varphi)\bigg|_0^{\pi/2}=\boxed{9\pi}$$

Calculation Notes: First, we simplified (computing the dot product and then canceling terms). Next, after changing the order of integration, we note that $\sin(\theta)$ integrated over the interval $[0,2\pi]$ yields an answer of 0. Then, the inner integral is constant with respect to theta, so we just multiply by the length of the interval (i.e., 2π). Finally, the last integral is done with a simple substitution: $u=\sin(\varphi)$ so $du=\cos(\varphi)\,d\varphi$.

Therefore, both the flux and line integral calculations yield the same answer (as they must).