Examples of Centroids

Center of mass for a lamina in the plane:

$$\text{mass:} \quad m = \iint_R \rho\, dA \qquad \text{moments:} \quad M_y = \iint_R x\rho\, dA \qquad M_x = \iint_R y\rho\, dA \qquad \text{center of mass:} \quad (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

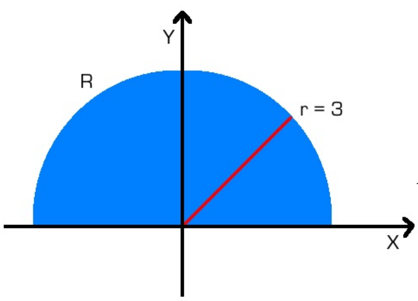
Center of mass for a solid in 3-space:

$$\text{mass:} \quad m = \iiint_E \rho\, dV \qquad \text{moments:} \quad M_{yz} = \iiint_E x\rho\, dV \qquad M_{xz} = \iiint_E y\rho\, dV \qquad M_{xy} = \iiint_E z\rho\, dV$$

$$\text{center of mass:} \quad (\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right)$$

Here we present a few examples of centroids. Recall that a center of mass of a region with constant density (i.e. $\rho = 1$) is called a *centroid*. In other words, the centroid of a region is its geometric center. First, we have two examples of centroids in $\mathbb{R}^2$. Then we present two examples of centroids of solid regions in $\mathbb{R}^3$.

Example: Find the centroid of $R = \{(x, y) \mid x^2 + y^2 \leq 9 \text{ and } y \geq 0\}$.



$$m = \iint\limits_R 1\, dA = \frac{1}{2}\pi 3^2 = \frac{9\pi}{2}.$$

We see that, by symmetry, $\bar{x} = 0$. So we are left to compute $\bar{y}$. First, we find the moment about the x-axis:

$$M_x = \iint\limits_R y\, dA = \int_0^\pi \int_0^3 r\sin(\theta) \cdot r\, dr d\theta = \int_0^\pi \sin(\theta)\, d\theta \int_0^3 r^2\, dr = 2 \cdot \frac{1}{3}\left[3^3 - 0\right] = 18.$$

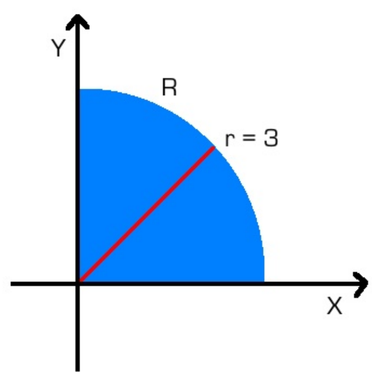
Hence $\bar{y} = \dfrac{18}{9\pi/2} = \dfrac{4}{\pi}$. We conclude that $\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{4}{\pi}\right) \text{ is the centroid of } R.}$

Example: Find the centroid of $R = \{(x, y) \mid x^2 + y^2 \leq 9 \text{ and } x, y \geq 0\}$.

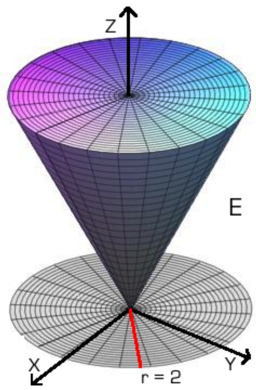
$$m = \iint\limits_R 1\, dA = \frac{1}{4}\pi 3^2 = \frac{9}{4}\pi.$$

$$M_y = \iint\limits_R x\, dA = \int_0^{\pi/2} \int_0^3 r\cos(\theta) \cdot r\, dr d\theta = \int_0^{\pi/2} \cos(\theta)\, d\theta \int_0^3 r^2\, dr = 1 \cdot \frac{1}{3}\left[3^3 - 0\right] = 9.$$

And so, we have that $\bar{x} = \dfrac{9}{9\pi/4} = \dfrac{4}{\pi}$. Notice that, by symmetry, we have $\bar{x} = \bar{y}$ and hence we may conclude that $\boxed{(\bar{x}, \bar{y}) = \left(\frac{4}{\pi}, \frac{4}{\pi}\right) \text{ is the centroid of } R.}$



Example: Find the centroid of the solid cone $E = \{(x,y,z) \mid \sqrt{x^2+y^2} \leq z \leq 2\}$.



$$m = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^2\int_r^2 r\,dzdrd\theta = \int_0^{2\pi} d\theta \cdot \int_0^2\int_r^2 dzdr = 2\pi\int_0^2 rz\Big|_r^2 dr$$

$$= 2\pi\int_0^2 2r - r^2\,dr = 2\pi\left[r^2 - \frac{1}{3}r^3\right]\Big|_0^2 = 2\pi\left[4 - \frac{8}{3}\right] = \frac{8}{3}\pi.$$

$$M_{z=0} = M_{xy} = \iiint_E z\,dV = \int_0^{2\pi}\int_0^2\int_r^2 zr\,dzdrd\theta = \int_0^{2\pi} d\theta \cdot \int_0^2 \frac{1}{2}z^2r\Big|_r^2 dr$$

$$= 2\pi\int_0^2 2r - \frac{1}{2}r^3 dr = 2\pi\left[r^2 - \frac{1}{8}r^4\right]\Big|_0^2 = 2\pi[4-2] = 4\pi.$$

$\bar{z} = \dfrac{4\pi}{8\pi/3} = \dfrac{3}{2}$and notice that $\bar{x} = \bar{y} = 0$ by symmetry so $\boxed{(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{3}{2}\right) \text{ is the centroid of } E.}$

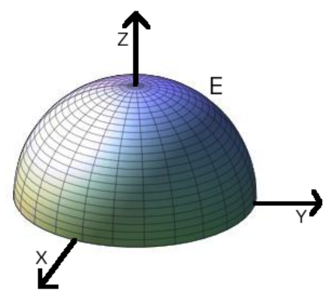
Example: Centroid of the upper-half sphere $E = \{(x,y,z) \mid 0 \leq z \leq \sqrt{4-x^2-y^2}\}$.

$$m = \iiint_E 1\,dV = \frac{1}{2}\left[\frac{4}{3}\pi \cdot 2^3\right] = \frac{16}{3}\pi.$$

Again, $\bar{x} = \bar{y} = 0$ by symmetry.

$$M_{xy} = \iiint_E z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho\cos(\phi)\rho^2\sin(\phi)\,d\rho d\phi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin(\phi)\cos(\phi)\,d\phi \int_0^2 \rho^3\,d\rho = 2\pi \cdot \frac{1}{2}\sin^2(\phi)\Big|_0^{\pi/2} \cdot \frac{1}{4}[2^4 - 0^4] = 4\pi$$



So, we have that $\bar{z} = \dfrac{4\pi}{16\pi/3} = \dfrac{3}{4}$ and as a result we conclude that the centroid of E is $\boxed{(\bar{x},\bar{y},\bar{z}) = \left(0,0,\frac{3}{4}\right).}$