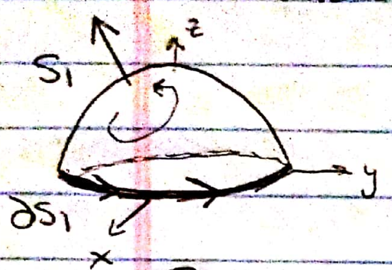
Ex: Verify Stokes' Thm (example from 8am - unfinished) on Monday

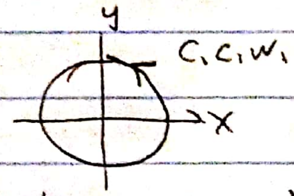
$S_1: x^2+y^2+z^2=9,\ z\geq 0$ orient upward

$\vec{F} = \langle z, x, x^2+y^2+z^2\rangle$

We want to show: $\underbrace{\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma}_{\text{II}} = \underbrace{\int_{\partial S_1} \vec{F}\cdot d\vec{r}}_{\text{I}}$



$\partial S_1: x^2+y^2+0^2=9$ & $z=0$



I. param ∂S_1: $\vec{r}(t) = \langle \overbrace{3\cos(t)}^{x}, 3\sin(t), \overbrace{0}^{z}\rangle$ $\quad 0\leq t\leq 2\pi$ $\quad$ C.C.W. ✓

derivative: $\vec{r}\,'(t) = \langle -3\sin(t), 3\cos(t), 0\rangle$

$$\int_{\partial S_1} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle \underbrace{0}_{z}, \underbrace{3\cos(t)}_{x}, \underbrace{9}_{x^2+y^2+z^2}\rangle \cdot \langle -3\sin(t), 3\cos(t), 0\rangle\,dt$$

$$= \int_0^{2\pi} 9\cos^2(t)\,dt = \int_0^{2\pi} \tfrac{9}{2}(1+\cos(2t))\,dt = \tfrac{9}{2}\cdot(2\pi+0) = \boxed{9\pi}$$

II. $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ z & x & x^2+y^2+z^2 \end{vmatrix} = \langle 2y-0, -(2x-1), 1-0\rangle$

$= \langle 2y, 1-2x, 1\rangle$

param S_1: (use sph. coords) $\quad \vec{r}(\varphi,\theta) = \langle \overbrace{3\cos\theta\sin\varphi}^{x}, \overbrace{3\sin\theta\sin\varphi}^{y}, \underbrace{3\cos\varphi}_{z}\rangle$

$\underbrace{0\leq\varphi\leq\tfrac{\pi}{2}}_{\text{upper half}}$ & $0\leq\theta\leq 2\pi$

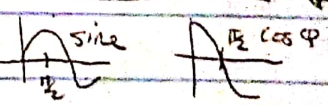
derivative:

$\vec{r}_\varphi = \langle 3\cos\theta\cos\varphi, 3\sin\theta\cos\varphi, -3\sin\varphi\rangle$

$\times\ \vec{r}_\theta = \langle -3\sin\theta\sin\varphi, 3\cos\theta\sin\varphi, 0\rangle$

$\vec{r}_\varphi\times\vec{r}_\theta = \langle 9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, \underbrace{9\sin\varphi\cos\varphi}_{\geq 0}\rangle$ up ✓

Recall: $\vec{n}\,d\sigma = \dfrac{\pm\vec{r}_\varphi\times\vec{r}_\theta}{\cancel{|\vec{r}_\varphi\times\vec{r}_\theta|}}\cancel{|\vec{r}_\varphi\times\vec{r}_\theta|}\,dA$



$$\iint_{S_1} (\nabla\times\vec{F})\cdot\vec{n}\,d\sigma = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \langle \underbrace{6\sin\theta\sin\varphi}_{2y}, \underbrace{1-6\cos\theta\sin\varphi}_{1-2x}, 1\rangle \cdot \langle 9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, 9\sin\varphi\cos\varphi\rangle\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cancel{54\sin^3\varphi\sin\theta\cos\theta} + 9\sin^2\varphi\underbrace{\sin\theta}_{\substack{\text{after int. this contributes } 0\\ 0\leq\theta\leq 2\pi}} - \cancel{54\sin^3\varphi\sin\theta\cos\theta} + 9\sin\varphi\cos\varphi\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 9\sin\varphi\cos\varphi\,d\varphi\,d\theta = \int_0^{2\pi} d\theta\cdot\int_0^{\frac{\pi}{2}} 9\sin\varphi\cos\varphi\,d\varphi = 2\pi\,\tfrac{9}{2}\sin^2\varphi\Big|_0^{\frac{\pi}{2}} = \boxed{9\pi}$$ ✓