It can be extremely useful to notice that if we preform an elementary row operation on A, then the linear relationships between columns of A will not change.
Specifically...Suppose that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are columns of A, and suppose that we preform some row operation and get A' whose corresponding columns are $\mathbf{a}'$, $\mathbf{b}'$, and $\mathbf{c}'$. Then it turns out that:

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0} \qquad \text{if and only if} \qquad x\mathbf{a}' + y\mathbf{b}' + z\mathbf{c}' = \mathbf{0}$$

where x, y, and z are some real numbers.

This also holds for bigger or smaller collections of columns. Why? Essentially, since we are preforming **row** operations, the relationships between **columns** should be unaffected – we aren't "mixing" the columns together. For example:

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 2 & 1 & 3 \end{bmatrix} \underset{\sim}{R1\leftrightarrow R2} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{bmatrix} \underset{\sim}{2R1+R3} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \underset{\sim}{-R2+R3} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{-1\times R1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice in the RREF (on the far right) we have that the final column is 2 times the first column plus -1 times the second column.

$$2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

So this must be true for all of the other matrices as well. In particular, we have that the third column of the original matrix is 2 times the first column plus -1 times the second column:

$$2\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

Why is this important? Well, first, Gauss-Jordan elimination typically requires a lot of computation. This correspondence gives you a way to check that you row-reduced correctly! We will see other applications of this correspondence later in the course.

Another example: Let A be a 3x3 matrix whose RREF is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Suppose that we know the first two column of A is $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ and the second is $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$. Then, by the linear correspondence, we know that the third column must be $-1 \cdot \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$. Therefore, the mystery matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.