Math 2240 | Determinants – ANSWER KEY | Project #3

I. From section 3.5, do problems 11, 37, and 38 – sketch a nice picture to help explain the system of equations in #38.

3.5 #11 Show that $|\text{adj}(A)| = |A|^{n-1}$ for every $n \times n$ matrix A.

We know that $A^{-1} = \dfrac{1}{\det(A)}\text{adj}(A)$ so that $|A^{-1}| = \left|\dfrac{1}{\det(A)}\text{adj}(A)\right|$ But $|A^{-1}| = |A|^{-1}$ and $\left|\dfrac{1}{\det(A)}\text{adj}(A)\right| = \left(\dfrac{1}{\det(A)}\right)^n |\text{adj}(A)|$ (since $|cB| = c^n|B|$ for any $n \times n$ matrix B). Thus we have that $|A|^{-1} = |A|^{-n}|\text{adj}(A)|$ multiplying both sides by $|A|^n$ we obtain $|A|^{n-1} = |\text{adj}(A)|$.

3.5 #37 Use Cramer's rule to solve:

$$\begin{array}{rcrcl} kx & + & (1-k)y & = & 1 \\ (1-k)x & + & ky & = & 3 \end{array}$$

Then determine which values of k make this system inconsistent.

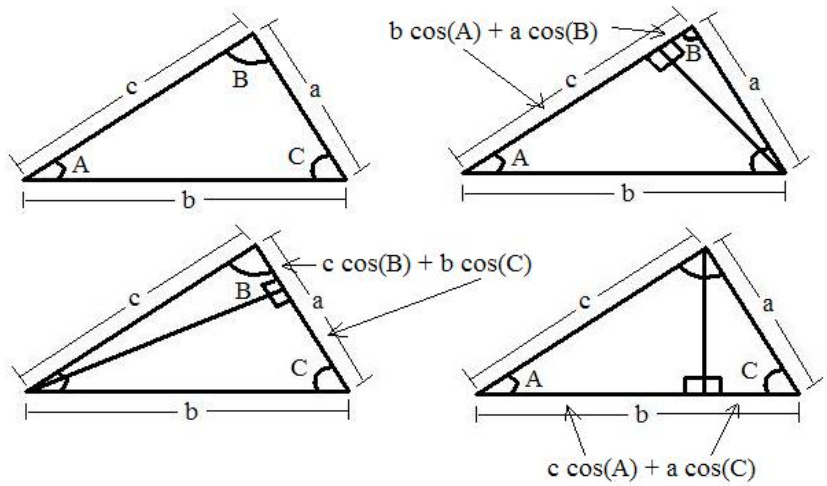
Our coefficient matrix is $A = \begin{bmatrix} k & 1-k \\ k & 1-k \end{bmatrix}$. $|A| = k^2 - (1-k)^2 = k^2 - k^2 + 2k - 1 = 2k - 1$. Replacing the first column of A with our constant terms we get: $\begin{bmatrix} 1 & 1-k \\ 3 & k \end{bmatrix}$ whose determinant is $k - 3(1-k) = 4k - 3$. Thus we have that $x = \dfrac{4k-3}{2k-1}$. Replacing the second column of A with our constant terms we get: $\begin{bmatrix} k & 1 \\ 1-k & 3 \end{bmatrix}$ whose determinant is $3k - (1-k) = 4k - 1$. Thus we have that $y = \dfrac{4k-1}{2k-1}$. Of course Cramer's rule fails if $k = 1/2$. Consider this case ($k = 1/2$):

$$\begin{array}{rcrcl} (1/2)x & + & (1/2)y & = & 1 \\ (1/2)x & + & (1/2)y & = & 3 \end{array}$$

Which says that $2 = x + y = 6$ which is impossible, so there is no solution when $k = 1/2$.

3.5 #38 Explain why the following equations hold (a, b, c and A, B, C refer to the length of sides and angles of the triangle pictured below). Use Cramer's rule to solve the following equations and thus establish the "Law of Cosines": $c^2 = a^2 + b^2 - 2ab\cos(C)$.

$$\begin{array}{rcrcrcl} & & c\cos(B) & + & b\cos(C) & = & a \\ c\cos(A) & & & + & a\cos(C) & = & b \\ b\cos(B) & + & a\cos(B) & & & = & c \end{array}$$



The diagrams above should adequately explain why the equations hold. Treating $\cos(A)$, $\cos(B)$, and $\cos(C)$ as our unknowns, we have the following coefficient matrix:

$$A = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$$

whose determinant is $|A| = 0 - c \det \begin{bmatrix} c & a \\ b & 0 \end{bmatrix} + b \det \begin{bmatrix} c & 0 \\ b & a \end{bmatrix} = -c(-ab) + b(ac) = 2abc.$

Next, replacing the third column of A with our constant terms, we get:

$$B = \begin{bmatrix} 0 & c & a \\ c & 0 & b \\ b & a & c \end{bmatrix}$$

whose determinant is $|B| = 0 - c \det \begin{bmatrix} c & b \\ b & c \end{bmatrix} + a \det \begin{bmatrix} c & 0 \\ b & a \end{bmatrix} = -c(c^2 - b^2) + a(ac) = c(a^2 + b^2 - c^2).$

Thus the third variable (i.e. $\cos(C)$) is equal to $\dfrac{c(a^2+b^2-c^2)}{2abc} = \dfrac{a^2+b^2-c^2}{2ab}$. Therefore, $2ab\cos(C) = a^2 + b^2 - c^2$ which gives us $c^2 = a^2 + b^2 - ab\cos(C)$ (the Law of Cosines).

II. An $n \times n$ matrix N is said to be *nilpotent* if $N^k = 0_{n\times n}$ for some positive integer k.

(a) Verify that $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent.

$$B^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

since $B^3 = 0$, B is nilpotent.

(b) Use the determinant to prove that nilpotent matrices are singular.

If N is nilpotent, $N^k = 0$ for some $k > 0$. Therefore, $|N|^k = |N^k| = |0| = 0$. But if $x^k = 0$, then $x = 0$. So we can conclude $|N| = 0$ and thus nilpotent matrices are not invertible.

(c) Let $\mathbf{v}$ be an eigenvector with eigenvalue λ for a matrix A. By definition, $A\mathbf{v} = \lambda\mathbf{v}$. What is $A^2\mathbf{v} =$? or in general, for $\ell \geq 0$, what is $A^\ell\mathbf{v} =$?

Notice that $A^2\mathbf{v} = AA\mathbf{v} = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda\lambda\mathbf{v} = \lambda^2\mathbf{v}$.
In general, $A^\ell\mathbf{v} = AA^{\ell-1}\mathbf{v} = A(\lambda^{\ell-1}\mathbf{v}) = \lambda^{\ell-1}A\mathbf{v} = \lambda^{\ell-1}\lambda\mathbf{v} = \lambda^\ell\mathbf{v}$.

(d) Find the eigenvalues and eigenvectors of the matrix B (from part (a)).

$$\det(\lambda I_3 - B) = \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda^3 = 0$$

Therefore, the only eigenvalue of B is $\lambda = 0$.
Plugging $\lambda = 0$ into $\lambda I_3 - B$ we get $-B$, so we need to solve the system $-B\mathbf{v} = \mathbf{0}$ to find eigenvectors.

$$[-B : \mathbf{0}] = \begin{bmatrix} 0 & -1 & 0 & : & 0 \\ 0 & 0 & -1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

If we label our variables v_1, v_2, v_3, we see that the above system says that $-v_2 = 0$ and $-v_3 = 0$. Therefore, $v_1 = t$ is free. We get that

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t \quad (t \neq 0) \quad \text{are the eigenvectors for } B.$$

(e) Prove that the only eigenvalue of a nilpotent matrix is zero (*Hint:* use part (c) and $N^k = 0$).

Suppose that N is nilpotent so that $N^k = 0$ ($k > 0$). Let $\mathbf{v} \neq \mathbf{0}$ be an eigenvector for N with eigenvalue λ therefore $N\mathbf{v} = \lambda\mathbf{v}$. By part (c) we have that $N^k\mathbf{v} = \lambda^k\mathbf{v}$. But $N^k = 0$ so that $N^k\mathbf{v} = 0\mathbf{v} = \mathbf{0}$. Therefore, $\lambda^k\mathbf{v} = \mathbf{0}$. Recall that $\mathbf{v}$ is an eigenvector so that (by definition) $\mathbf{v} \neq \mathbf{0}$, so we must conclude that $\lambda^k = 0$ hence $\lambda = 0$.