ANSWER KEY

I. From section 4.8, do problems 38 and 40 – Use Maple to plot these equations in both the original and rotated coordinates.
Note: Use `implicitplot` to plot these conic sections. For example:

[> with(plots) :

[> $\text{implicitplot}\left(x^2 + \frac{y^2}{4} = 1, x = -2..2, y = -2..2, \text{scaling} = \text{constrained}\right);$

Answer: Skip – see a previous posted example for how to do this in Maple.

II. Let U and W be subspaces of a vector space V.

(a) Show that $U \cap W = \{v \in V \mid v \in U \text{ and } v \in W\}$ is a subspace of V.

Vectors in U and W are vectors in V since U and W are subspaces – thus also subsets of V. Therefore, $U \cap W \subset V$.

Notice that $0 \in U$ and $0 \in W$ (since they are subspaces). Therefore, $0 \in U \cap W$ (so that $U \cap W$ is non-empty).

Let $u, v \in U \cap W$. This means that $u, v \in U$ and $u, v \in W$. But U and W are subspaces so that $u + v \in U$ and $u + v \in W$. Therefore, $u + v \in U \cap W$ (so that $U \cap W$ is closed under vector addition).

Let $u \in U \cap W$ and $c \in \mathbb{R}$. This means that $u \in U$ and $v \in W$ and thus $cu \in U$ and $cu \in W$ (since U and W are subspaces). Therefore, $cu \in U \cap W$ (so that $U \cap W$ is closed under scalar multiplication).

Therefore, $U \cap W$ is a subspace of V.

(b) Show that $U + W = \{u + w \mid u \in U \text{ and } w \in W\}$ is a subspace of V.

Vectors in U and W are vectors in V since U and W are subspaces – so sums of vectors in U and W are vectors in V. Therefore, $U + W \subset V$.

Notice that $0 \in U$ and $0 \in W$ (since they are subspaces). Therefore, $0 = 0 + 0 \in U + W$ (so that $U + W$ is non-empty).

Let $v_1, v_2 \in U + W$. This means that $v_i = u_i + w_i$ for some $u_i \in U$ and $w_i \in W$. However, U and W are subspaces, so $u_1 + u_2 \in U$ and $w_1 + w_2 \in W$. Therefore, $v_1 + v_2 = (u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2) \in U + W$ (so that $U + W$ is closed under vector addition).

Let $v \in U + W$ and $c \in \mathbb{R}$. This means that $v = u + w$ for some $u \in U$ and $w \in W$. However, U and W are subspaces, so $cu \in U$ and $cw \in W$. Therefore, $cv = c(u + w) = (cu) + (cw) \in U + W$ (so that $U + W$ is closed under scalar multiplication).

Therefore, $U + W$ is a subspace of V.

BONUS: Prove that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$

Let $\beta = \{v_1, \ldots, v_k\}$ be a basis for $U \cap W$. So β is a linearly independent subset of both U and W. Every linearly independent subset can be extended to a basis. Let $\gamma = \{v_1, \ldots, v_k, u_1, \ldots, u_\ell\}$ be such a basis for U and $\delta = \{v_1, \ldots, v_k, w_1, \ldots, w_m\}$ such a basis for W.

I claim that $\tau = \gamma \cup \delta = \{v_1, \ldots, v_k, u_1, \ldots, u_\ell, w_1, \ldots, w_m\}$ is a basis for $U + W$.

[*Note*: τ has $k+\ell+m$ elements since $u_i \neq w_j$ because if $u_i = w_j$ then $u_i = w_j \in U \cap W$ and thus we found an element independent of $v_1, \ldots, v_k$ in $U \cap W$ contradicting the fact that β is a basis for $U \cap W$.]

Notice that τ spans $U + W$. Let $v \in U + W$ so that $v = u + w$ for some $u \in U$ and $w \in W$. But $u = \sum a_i v_i + \sum b_j u_j$ because γ is a basis for U and $w = \sum c_i v_i + \sum d_j w_j$ because δ is a basis for W. Therefore, $v = u + w$ is a linear combination of the v_i's, u_i's, and w_i's so that τ spans $U + W$.

Finally, suppose that $\sum a_i v_i + \sum b_j u_j + \sum c_n w_n = 0$. Then $v' = \sum a_i v_i + \sum b_j u_j = -\sum c_n w_n$ which is an element of U (look at the left hand side) and an element of W (look at the right hand side). Therefore, $v' = \sum d_n v_n$ (since $v' \in U \cap W$ and $\beta = \{v_1, \ldots, v_k\}$ is a basis). Therefore, $0 = v' - v' = \sum d_i v_i + \sum c_j w_j$ and therefore, $d_i = 0$ and $c_j = 0$ for all i, j's (since δ is linearly independent). So we now have that $v' = \sum 0 w_j = 0$. So that $0 = \sum a_i v_i + \sum b_j u_j$ and thus $a_i = 0$ and $b_j = 0$ for all i, j's (since γ is linearly independent). Therefore, τ is linearly indepedent.

We have shown that τ is a basis for $U+W$. Notice we have the formula: $|\gamma|+|\delta|-|\beta| = (k+\ell)+(k+m)-k = k+\ell+m = |\tau|$ which says that $\dim(U)+\dim(W)-\dim(U\cap W) = \dim(U + W)$.

III. Consider $V = P_3$ (polynomials of degree 3 and less). Let $U = \{f(x) \in P_3 \,|\, f(0) = 0\}$ and let $W = \{f(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \,|\, a_3 + a_2 + a_1 + a_0 = 0\}$.

(a) Show that U and W are subspaces.

Notice that $0(0) = 0$ so that $0 \in U$. Also, $f, g \in U$ implies that $f(0) = 0$ and $g(0) = 0$ so that $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$. Thus $f + g \in U$. Finally, $f \in U$ and $c \in \mathbb{R}$ then $(cf)(0) = cf(0) = c0 = 0$ so that $cf \in U$. Therefore, U is a subspace of P_3 (non-empty + closed under vector addition + closed under scalar multiplication).

First, notice that $0+0+0+0 = 0$ so that $0 \in W$ (the sum of the zero polynomials coefficients is zero). Let $f, g \in W$ then $f(t) = a_3t^3 + a_2t^2 + a_1t + a_0$ where $a_4 +$

$a_3+a_2+a_1+a_0=0$ and $g(t)=b_3t^3+b_2t^2+b_1t+b_0$ where $b_4+b_3+b_2+b_1+b_0=0$. Therefore, $(f+g)(t)=(a_3+b_3)t^3+(a_2+b_2)t^2+(a_1+b_1)t+(a_0+b_0)$ and $(a_3+b_3)+(a_2+b_2)+(a_1+b_1)+(a_0+b_0)=(a_4+a_3+a_2+a_1+a_0)+(b_4+b_3+b_2+b_1+b_0)=0+0=0$ so that $f+g\in W$. Likewise if $c\in\mathbb{R}$ then $ca_4+ca_3+ca_2+ca_1+ca_0=c(a_4+a_3+a_2+a_1+a_0)=c0=0$ so that $cf\in W$. Therefore, W is a subspace of P_3.

(b) Show that $\beta=\{t,t^2,t^3\}$ is a basis for U.

Obviously $\beta\subset U$ (for each element, plug in zero and you get zero).
If $at+bt^2+ct^3=0$ then $at+bt^2+ct^3=0t^3+0t^2+0t+0\cdot 1$ so that $a=b=c=0$. Thus β is linearly independent.
Let $f\in U$. Then $f(t)=a_3t^3+a_2t^2+a_1t+a_0$ and $a_0=f(0)=0$. So we have that $f(t)=a_3t^3+a_2t^2+a_1t\in\text{span}(\beta)$. Therefore, β spans U. Thus β is a basis for U ($\dim(U)=3$).
Alternate proof: $1\notin U$ since if $h(t)=1$ then $h(0)=1\neq 0$. So $U\neq P_3$. Therefore, $\dim(U)<\dim(P_3)=4$. But β is a subset of the standard basis for P_3 so it is linearly independent. Thus U contains a linearly independent set of size 3. Thus $\dim(U)\geq 3$. So $\dim(U)=3$. Now β is a linearly independent set with $3=\dim(U)$ vectors so β automatically spans (so it is a basis for U).

(c) Find a basis W (remember to show that your basis *is* a basis).

Notice that W contains t^3-1, t^2-1, and $t-1$ (since their coefficients sum to $1-1=0$). Also, $a(t^3-1)+b(t^2-1)+c(t-1)=0$ implies that $at^3+bt^2+ct-(a+b+c)=0$ so that $a=b=c=0$. Therefore, $\gamma=\{t^3-1,t^2-1,t-1\}$ is a linearly independent subset of W (from this we also have that $\dim(W)\geq 3$). Notice that $W\neq P_3$ since $1\notin W$ (1's coefficients sum to 1 not 0). Therefore, $\dim(W)<\dim(P_3)=4$. Thus $\dim(W)=3$ and so γ automatically spans (so it is a basis for W).

(d) Find a basis for $U\cap W$.

Let $f\in U\cap W$ and $f(t)=a_3t^3+a_2t^2+a_1t+a_0$. $f\in U$ implies that $a_0=f(0)=0$ and $f\in W$ implies that $a_3+a_2+a_1+a_0=0$ so that $a_3+a_2+a_1=0$ and thus $a_3=-(a_2+a_1)$. So $f(t)=-(a_2+a_1)t^3+a_2t^2+a_1t$ If $a_1=0$, we need $a_3=-a_2$ and if $a_2=0$, then $a_3=-a_1$. Thus $-t^3+t^2$ and $-t^3+t$ are elements of $U\cap W$. Notice that $a(-t^3+t^2)+b(-t^3+t)=0$ implies that $-(a+b)t^3+at^2+bt=0$ so that $a=b=0$. Therefore, $\delta=\{-t^3+t^2,-t^3+t\}$ is a linearly independent subset of $U\cap W$. Moreover, we know that $f(t)=-(a_2+a_1)t^3+a_2t^2+a_1t=a_2(-t^3+t^2)+a_1(-t^3+t)$ so δ spans $U\cap W$ and so it is a basis.

(e) Show that $U+W=P_3$.

Notice that $\dim(U)=3$ and $t-1=0+(t-1)\in U+W$ but $t-1\notin U$ (since $0-1=-1\neq 0$). Therefore, $U+W$ is larger than U so that $\dim(U+W)>3$ which implies that $\dim(U+W)=4=\dim(P_3)$. Therefore, $U+W=P_3$.

Alternatively, notice that $f(t) = a_3t^3 + a_2t^2 + a_1t + a_0 = (a_3 + a_2 + a_1 + a_0)t^3 + (-a_2 - a_1 - a_0)t^3 + a_2t^2 + a_1t + a_0$ which is a sum of an element of U and an element of W. Therefore, every element in P_3 can be expressed as the sum of an element in U and an element of W so that $P_3 = U + W$.

Or (using the bonus problem) $\dim(U) + \dim(W) - \dim(U \cap W) = 3 + 3 - 2 = 4 = \dim(U + W)$ so $U + W = P_3$.