I. Let $T : P_2 \to P_2$ be defined by $T(f(t)) = f''(t) + 2f'(t) + f(t)$. Also, let $\alpha = \{1, t, t^2\}$ (the standard basis). Note: T is a linear transformation.

(a) Determine $[T]_\alpha$.

- $T(1) = 0 + 2(0) + 1 = 1 \implies [T(1)]_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $T(t) = 0 + 2(1) + t = 2 + t \implies [T(t)]_\alpha = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
- $T(t^2) = 2 + 2(2t) + t^2 = 2 + 4t + t^2 \implies [T(t)]_\alpha = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

Therefore, $[T]_\alpha = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Find bases for both the Kernel and Range of T. In addition, find the nullity and rank of T. Is T invertible?

Notice that $[T]_\alpha \sim I_3$ (the matrix of T row reduces to the identity). Thus the nullspace is the zero subspace and the column space is the everything. Therefore, $\text{Ker}(T) = \{0\}$ (T is one-to-one) and $\text{Range}(T) = P_2$ (T is onto). Therefore, $\phi = \{\}$ (the empty set) is a basis for the kernel and $\alpha = \{1, t, t^2\}$ is a basis for the range. Alternatively, we could use the "columns" of the matrix to find a basis for the range. This would give us, $\{1, 2 + t, 2 + 4t + t^2\}$ which is also a basis for $\text{Range}(T) = P_2$. Since T is one-to-one and onto, T is invertible...OR...Since $([T]_\alpha)^{-1}$ exists, so does T^{-1}.

(c) Find the eigenvalues of T. Find the cooresponding eigenvectors.

Using $[T]_\alpha$, it is easy to see that the characteristic polynomial of T is $f(t) = (t-1)^3$. So $\lambda = 1$ is the only eigenvalue of T and its algebraic multiplicity is 3.
Let's find eigenvectors for $[T]_\alpha$ and $\lambda = 1$:

$$[1I_3 - [T]_\alpha \ : \ \mathbf{0}] = \begin{bmatrix} 0 & -2 & -2 & : & 0 \\ 0 & 0 & -4 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \overset{\text{G.E.}}{\sim} \begin{bmatrix} 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Labeling variables x_1, x_2, and x_3, we see that $x_2 = 0$, $x_3 = 0$, and x_1 is free. Therefore, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s$ is an eigenvector for all $s \neq 0$. Remember that we are looking for eigenvectors for T (not $[T]_\alpha$). The vectors we just found are coordinate vectors for the polynomials $f(t) = s + 0t + 0t^2$. So the eigenvectors for T (with eigenvalue $\lambda = 1$) are exactly the non-zero constant polynomials.

(d) Determine all geometric and algebraic multiplicities. Is T diagonalizable?

"NO" T is not diagonalizable.
We found that the only eigenvalue of T is $\lambda = 1$. Its algebraic multiplicity is 3 and geometric multiplicity is 1. Since $1 < 3$ (T doesn't have "enough" eigenvectors), T is not diagonalizable.

II. Let $A = \begin{bmatrix} 3 & -2 & 3 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

(a) Find eigenvectors and eigenvalues for A.

(Expand along the bottom row.)

$$\det(tI_3 - A) = \det\left(\begin{bmatrix} t-3 & 2 & -3 \\ 1 & t-2 & 1 \\ 0 & 0 & t \end{bmatrix}\right) = t \cdot \det\left(\begin{bmatrix} t-3 & 2 \\ 1 & t-2 \end{bmatrix}\right)$$
$$= t[(t-3)(t-2) - 2] = t(t^2 - 5t + 4) = t(t-4)(t-1)$$

So the eigenvalues of A are $\lambda = 0, 4, 1$ (each with multiplicity 1). Therefore, A is diagonalizable.
Let's find eigenvectors.

$\lambda = 0 \,:\, [0I_3 - A \,:\, \mathbf{0}] \begin{bmatrix} -3 & 2 & -3 & : & 0 \\ 1 & -2 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \overset{\text{G.E.}}{\sim} \begin{bmatrix} 1 & 0 & 1 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$ Labeling variables x_1, x_2, and x_3, we see that $x_1 + x_3 = 0$, $x_2 = 0$, and x_3 is free. Therefore, we get $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$ is an eigenvector with eigenvalue $\lambda = 0$ for all $t \neq 0$.

$\lambda = 4 \,:\, [4I_3 - A \,:\, \mathbf{0}] \begin{bmatrix} 1 & 2 & -3 & : & 0 \\ 1 & 2 & 1 & : & 0 \\ 0 & 0 & 4 & : & 0 \end{bmatrix} \overset{\text{G.E.}}{\sim} \begin{bmatrix} 1 & 2 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$ Labeling variables x_1, x_2, and x_3, we see that $x_1 + 2x_2 = 0$, $x_3 = 0$, and x_2 is free. Therefore, we get $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} t$ is an eigenvector with eigenvalue $\lambda = 4$ for all $t \neq 0$.

$\lambda = 1 \,:\, [1I_3 - A \,:\, \mathbf{0}] \begin{bmatrix} -2 & 2 & -3 & : & 0 \\ 1 & -1 & 1 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix} \overset{\text{G.E.}}{\sim} \begin{bmatrix} 1 & -1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$ Labeling variables x_1, x_2, and x_3, we see that $x_1 - x_2 = 0$, $x_3 = 0$, and x_2 is free. Therefore, we get $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} t$ is an eigenvector with eigenvalue $\lambda = 1$ for all $t \neq 0$.

(b) Find a matrix, P, which diagonalizes A.

Therefore, $P = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ diagonalizes A

with corresponding diagonal matrix $D = P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) Use part (b) to find $B = \sqrt{A}$ (that is: $B^2 = A$).

Since $D = P^{-1}AP$, we have $A = PDP^{-1}$ and so

$$\sqrt{A} = P\sqrt{D}P^{-1} = \begin{bmatrix} -1 & -2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{0} & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{1} \end{bmatrix} \frac{1}{3} \begin{bmatrix} 0 & 0 & 3 \\ -1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 & -2 & 5 \\ -1 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We can double check and find that $\sqrt{A} \cdot \sqrt{A} = A$ ✓

III. Let A be an $n \times n$ matrix.

(a) Show that A is non-singular (i.e. invertible) if and only if $\lambda = 0$ is not an eigenvalue of A.

There are a million ways to prove this. Here are two:

- Let $f(t) = \det(tI - A)$ (the characteristic polynomial). $\lambda = 0$ is an eigenvalue if and only if $f(\lambda) = f(0) = 0 \Leftrightarrow (-1)^n\det(A) = \det(-A) = f(0) = 0$. So $\lambda = 0$ if and only if $\det(A) = 0$ (i.e. A is not invertible).
- Let $\lambda = 0$ be an eigenvalue. So there must be an eigevector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$. This means that the homogeneous system of equations $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution so that nullity$(A) \neq 0$ and thus A^{-1} does not exist. Conversely, if A^{-1} does not exist, then nullity$(A) \neq 0$ and so there is a non-trivial solution, $\mathbf{x} \neq \mathbf{0}$, for $A\mathbf{x} = \mathbf{0}$. This non-trivial solution is an eigenvector with eigenvalue 0.

(b) Show that A and A^{-1} have the same eigenvectors.

We are implicitly assuming that A^{-1} exists. Therefore, by part (a), the eigenvalues of A are non-zero.

Let $\mathbf{v} \neq \mathbf{0}$ be an eigenvector with eigenvalue λ ($\lambda \neq 0$ since A is invertible). Therefore, $A\mathbf{v} = \lambda\mathbf{v}$. Mutliply both sides of this equation (on the left) by A^{-1} and get: $\mathbf{v} = I_n\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\lambda\mathbf{v}$. Therefore, $\lambda A^{-1}\mathbf{v} = \mathbf{v}$. Since $\lambda \neq 0$ we can divide both sides by λ and get $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$. Therefore, $\mathbf{v}$ is also an eigevector for A^{-1} with eigenvalue λ^{-1}.

Finally, replacing A with A^{-1} in the above argument, we see that if $\mathbf{v}$ is an eigenvector for A^{-1} with eigenvalue λ, then $\mathbf{v}$ is an eigenvector for A with eigenvalue λ^{-1}.

Therefore, A and A^{-1} have the same eigenvectors.

Note: Since A and A^{-1} have the same eigenvectors, we can conclude that A is diagonalizable if and only if A^{-1} is diagonalizable. Moreover, the same matrix will diagonalize both!

(c) How are the eigenvalues of A and A^{-1} related? Give an example. Then prove your conjectured relationship in general.

Our previous proof (from part (b)) shows that λ is an eigenvalue for A if and only if λ^{-1} is an eigenvalue for A^{-1}.

Example: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $D = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ so the eigenvalues of A are -1 and 3.
Notice that $A^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ and $D^{-1} = P^{-1}A^{-1}P = \begin{bmatrix} -1 & 0 \\ 0 & 1/3 \end{bmatrix}$ so the eigenvalues of A^{-1} are -1 and $1/3$.