Solving systems of linear equations lies at the heart of linear algebra. In high school we learn to solve systems in 2 or 3 variables using "elimination" and "substitution" of variables. In order to solve systems with a large number of variables we need to be more organized. The process of *Gauss-Jordan Elimination* gives us a systematic way of solving linear systems.

To solve a system of equations we first drop as many unncessary symbols as possible. This is done by constructing an *augmented matrix*.

Example:

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \quad \Longrightarrow \quad \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix}$$

Manipulating the system of equations corresponds to manipulating *rows* of the matrix. It turns out that we only need 3 types of operations to solve any linear system. We call these *elementary operations*.

Definition: Elementary Row Operations

	Effect on the matrix:		**Effect of the linear system:**
Type I	Swap Row i and Row j	$\Longleftrightarrow$	Interchange equation i and equation j (List the equations in a different order.)
Type II	Multiply Row i by c where $c \neq 0$	$\Longleftrightarrow$	Multiply both sides of equation i by a non-zero scalar c
Type III	Add c times Row i to Row j where c is any scalar	$\Longleftrightarrow$	Multiply both sides of equation i by c and add to equation j

If we can get matrix A from matrix B by performing a series of elementary row operations, then A and B are called **row equivalent matrices**.

Example: Type I — swap rows 1 and 3

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \quad \Longrightarrow \quad \begin{array}{rcrcrcr} -x & & & + & 4z & = & 7 \\ & & 5y & - & 6z & = & 0 \\ 2x & - & y & + & 3z & = & -1 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix} \underset{\sim}{R1 \leftrightarrow R3} \begin{bmatrix} -1 & 0 & 4 & : & 7 \\ 0 & 5 & -6 & : & 0 \\ 2 & -1 & 3 & : & -1 \end{bmatrix}$$

Example: Type II — scale row 3 by -2

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \quad \Longrightarrow \quad \begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ 2x & & & + & -8z & = & -14 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix} \underset{\sim}{-2 \times R3} \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ 2 & 0 & -8 & : & -14 \end{bmatrix}$$

Example: Type III — add 3 times row 3 to row 2

$$\begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ & & 5y & - & 6z & = & 0 \\ -x & & & + & 4z & = & 7 \end{array} \quad \Longrightarrow \quad \begin{array}{rcrcrcr} 2x & - & y & + & 3z & = & -1 \\ -3x & & 5y & + & 6z & = & 21 \\ -x & & & + & 4z & = & 7 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 3 & : & -1 \\ 0 & 5 & -6 & : & 0 \\ -1 & 0 & 4 & : & 7 \end{bmatrix} \underset{\sim}{3 \times R3 + R2} \begin{bmatrix} 2 & -1 & 3 & : & -1 \\ -3 & 5 & 6 & : & 21 \\ -1 & 0 & 4 & : & 7 \end{bmatrix}$$

It is important to notice several things about these operations. First, they are all reversible (that's why we want $c \neq 0$ in type II operations) — in fact the inverse of a type X operation is another type X operation. Next, these operations don't effect the set of solutions for the system — that is — row equivalent matrices represent systems with the same set of solutions. Finally, these are *row* operations — columns *never* interact with each other.

Doing operations blindly probably won't get us anywhere. What we want is to head towards some shape of equations which will let us read off the set of solutions. Thus the next few defintions.

Definition: A matrix is in **row echelon form** (or REF) if

- Each non-zero row is above all zero rows – that is – zero rows are "pushed" to the bottom.
- The leading entry of a row is *strictly* to the right of the leading entries of the rows above. (The leftmost non-zero entry of a row is called the "leading entry".)

If in addition...

- Each leading entry is "1". (*Note:* Our textbook says this is a requirement of REF.)
- Only zeros appear above (& below) a leading entry of a row.

then a matrix is in **reduced row echelon form** (or RREF).

Gauss-Jordan Elimination is an "algorithm" which given a matrix returns a row equivalent matrix in reduced row echelon form.

1. Determine the leftmost non-zero column. This is a **pivot column** and the topmost entry is a **pivot position**. If "0" is in this pivot position, swap (an unignored) row with the topmost row (use a Type I operation) so that there is a non-zero entry in the pivot position.
2. Add appropriate multiples of the topmost (unignored) row to the rows beneath it so that only "0" appears below the pivot (use several Type III operations).
3. Ignore the topmost (unignored) row. If any non-zero rows remain, go to step 1.

This part of Gauss-Jordan Elimination is called the **forward pass**. This part of the process will put our matrix in row echelon form (in my sense not our textbook's sense). Now let's finish Gauss-Jordan Elimination.

1. If necessary, scale the rightmost unfinished pivot to 1 (use a Type II operation).
2. Add appropriate multiples of the current pivot's row to rows above it so that only 0 appears above the current pivot (using several Type III operations).
3. The current pivot is now "finished". If any unfinished pivots remain, go to step 4.

This part of Gauss-Jordan Elimination is called the **backward pass**. It should be fairly obvious that this algorithm will terminate in finitely many steps. Also, only elementary row operations have been used. So we end up with a row equivalent matrix. A tedious and wordy proof shows that the resulting matrix is in reduced row echelon form.

Example: Let's solve the system
$$\begin{array}{rcrcr} x & + & 2y & = & 1 \\ 3x & + & 4y & = & -1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & : & 1 \\ 3 & 4 & : & -1 \end{bmatrix} \overset{-3\times R1+R2}{\sim} \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix}$$

The first non-zero column is just the first column. So the upper left hand corner is a pivot position. This position already has a non-zero entry so no swap is needed. The type III operation "-3 times row 1 added to row 2" clears the only position below the pivot, so after one operation we have finished with this pivot and can ignore row 1. $\begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix}$

Among the (unignored parts of) columns the leftmost non-zero column is the second column. So the "-2" sits in a pivot position. Since it's non-zero, no swap is needed. Also, there's nothing below it, so no type III operations are necessary. Thus we're done with this row and we can ignore it.

$$\begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix}$$

Nothing's left so we're done with the forward pass. $\begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix}$ is in row echelon form.

Next, we need to take the rightmost pivot (the "-2") and scale it to 1 then clear everything above it.

$$\begin{bmatrix} 1 & 2 & : & 1 \\ 0 & -2 & : & -4 \end{bmatrix} \underset{\sim}{\overset{-1/2\times R2}{}} \begin{bmatrix} 1 & 2 & : & 1 \\ 0 & 1 & : & 2 \end{bmatrix} \underset{\sim}{\overset{-2\times R2+R1}{}} \begin{bmatrix} 1 & 0 & : & -3 \\ 0 & 1 & : & 2 \end{bmatrix}$$

This "finishes" that pivot. The next rightmost pivot is the 1 in the upper left hand corner. But it's already scaled to 1 and has nothing above it, so it's finished as well. That takes care of all of the pivots so the backward pass is complete leaving our matrix in reduced row echelon form.

Finally, let translate the RREF matrix back into a system of equations. The (new equivalent) system is $\begin{array}{rcr} x & = & -3 \\ y & = & 2 \end{array}$. So the only solution for this system is $x = -3$ and $y = 2$.

Note: One can also solve a system quite easily once (just) the forward pass is complete. This is done using "back substitution". Notice that the system after the forward pass was $\begin{array}{rcrcr} x & + & 2y & = & 1 \\ & & -2y & = & -4 \end{array}$. So we have $-2y = -4$ thus $y = 2$. Substituting this back into the first equation we get $x+2(2) = 1$ so $x = -3$.

Example: Let's solve the system $\begin{array}{rcrcrcr} 2x & - & y & + & z & = & 0 \\ 4x & - & 2y & + & 2z & = & -1 \\ x & & & + & z & = & 1 \end{array}$

$$\begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 4 & -2 & 2 & : & 1 \\ 1 & 0 & 1 & : & -1 \end{bmatrix} \underset{\sim}{\overset{-2\times R1+R2}{}} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 1 & 0 & 1 & : & -1 \end{bmatrix} \underset{\sim}{\overset{-1/2\times R1+R3}{}} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 0 & 1/2 & 1/2 & : & -1 \end{bmatrix} \underset{\sim}{\overset{\text{Ignore } R1}{}}$$

$$\begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \\ 0 & 1/2 & 1/2 & : & -1 \end{bmatrix} \underset{\sim}{\overset{R2\leftrightarrow R3}{}} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \underset{\sim}{\overset{\text{Ignore } R2}{}} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \underset{\sim}{\overset{\text{Ignore } R3}{}}$$

which leaves us with nothing. So the forward pass is complete and $\begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$ is in REF.

$$\begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & -1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \underset{\sim}{\overset{1\times R3+R2}{}} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1/2 & 1/2 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \underset{\sim}{\overset{2\times R2}{}} \begin{bmatrix} 2 & -1 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \underset{\sim}{\overset{1\times R2+R1}{}}$$

$$\begin{bmatrix} 2 & 0 & 2 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{bmatrix} \underset{\sim}{\overset{1/2\times R1}{}} \begin{bmatrix} 1 & 0 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$$

This finishes the backward pass and our matrix is now in RREF. Our new system of equations is $\begin{array}{rcrcrcr} x & & & + & z & = & 0 \\ & & y & + & z & = & 0 \\ & & & & 0 & = & 1 \end{array}$. Of course $0 \neq 1$, so this is an **inconsistant** system — it has **no solutions**.

Note: If our only goal was to solve this system, we could have stopped after the very first operation (row number 2 already said "$0 = 1$").

Example: Let's solve the system
$$\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 3 \\ 4x &+& 5y &+& 6z &=& 9 \\ 7x &+& 8y &+& 9z &=& 15 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 4 & 5 & 6 & : & 9 \\ 7 & 8 & 9 & : & 15 \end{bmatrix} \underbrace{\;}_{-4\times R1+R2} \begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 0 & -3 & -6 & : & -3 \\ 7 & 8 & 9 & : & 15 \end{bmatrix} \underbrace{\;}_{-7\times R1+R3} \begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 0 & -3 & -6 & : & -3 \\ 0 & -6 & -12 & : & -6 \end{bmatrix} \underbrace{\;}_{-2\times R2+R3}$$

$$\begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 0 & -3 & -6 & : & -3 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \underbrace{\;}_{-1/3\times R2} \begin{bmatrix} 1 & 2 & 3 & : & 3 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \underbrace{\;}_{-2\times R2+R1} \begin{bmatrix} 1 & 0 & -1 & : & 1 \\ 0 & 1 & 2 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Now our matrix is in RREF. The new system of equations is $\begin{array}{rcrcl} x & & -z &=& 1 \\ & y &+ 2z &=& 1 \\ & & 0 &=& 0 \end{array}$. This is new — we don't have an equation of the form "$z = ...$" This is because z does not lie in a pivot column. So we can make z a "free variable." Let's relabel it $z = t$. Then we have $x - t = 1$, $y + 2t = 1$, and $z = t$. So $x = 1 + t$, $y = 1 - 2t$, and $z = t$ is a solution for any choice of t. In particular, $x = y = 1$ and $z = 0$ is a solution. But so is $x = 2$, $y = -1$, $z = 1$. In fact, there are infinitely many solutions.

Note: A system of linear equations will always have either one solution, infinitely many solutions, or no solution at all.

Multiple Systems: Gauss-Jordan can handle solving multiple systems at once, if these systems share the same coefficient matrix (the part of the matrix before the :'s).

Suppose we wanted to solve both $\begin{array}{rcrcr} x &+& 2y &=& 3 \\ 4x &+& 5y &=& 6 \\ 7x &+& 8y &=& 9 \end{array}$ and also $\begin{array}{rcrcr} x &+& 2y &=& 3 \\ 4x &+& 5y &=& 9 \\ 7x &+& 8y &=& 15 \end{array}$. These lead to the following augmented matrices: $\begin{bmatrix} 1 & 2 & : & 3 \\ 4 & 5 & : & 6 \\ 7 & 8 & : & 9 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & : & 3 \\ 4 & 5 & : & 9 \\ 7 & 8 & : & 15 \end{bmatrix}$. We can combine them together and get $\begin{bmatrix} 1 & 2 & : & 3 & 3 \\ 4 & 5 & : & 6 & 9 \\ 7 & 8 & : & 9 & 15 \end{bmatrix}$ which we already know has the RREF of $\begin{bmatrix} 1 & 0 & : & -1 & 1 \\ 0 & 1 & : & 2 & 1 \\ 0 & 0 & : & 0 & 0 \end{bmatrix}$ (from the last example — only the :'s have moved). This corresponds to the augmented matrices $\begin{bmatrix} 1 & 0 & : & -1 \\ 0 & 1 & : & 2 \\ 0 & 0 & : & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & : & 1 \\ 0 & 1 & : & 1 \\ 0 & 0 & : & 0 \end{bmatrix}$. These in turn tell us that the first system's solution is $x = -1$, $y = 2$ and the second system's solution is $x = 1$ and $y = 1$.

This works because we aren't mixing columns together (all operations are row operations). Also, notice that the same matrix can be interpreted in a number of ways. Before we had a single system in 3 variables and now we have 2 systems in 2 variables.

Homework Problems: For each of the following matrices perform Gauss-Jordan elimination. I have given the result of the performing the forward pass to help verify you're on the right track. Also, identify the pivots and pivot columns. Finally, interpret your matrices as a system or collection of systems of equations and note the corresponding solutions.

$$\begin{bmatrix} 1 & 5 \\ 2 & 2 \end{bmatrix} \Longrightarrow \text{forward pass} \Longrightarrow \begin{bmatrix} 1 & 5 \\ 0 & -8 \end{bmatrix} \qquad \begin{bmatrix} 2 & -4 & 0 & 4 \\ 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 4 & -8 & 1 & 6 \end{bmatrix} \Longrightarrow \text{forward pass} \Longrightarrow \begin{bmatrix} 2 & -4 & 0 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \Longrightarrow \text{forward pass} \Longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & -2 \\ 1 & -3 & -2 \end{bmatrix} \Longrightarrow \text{forward pass} \Longrightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

It can be extremely useful to notice that if we preform an elementary row operation on A, then the linear relationships between columns of A will not change.
Specifically... Suppose that $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are columns of A, and suppose that we perform some row operation and get A' whose corresponding columns are $\mathbf{a}'$, $\mathbf{b}'$, and $\mathbf{c}'$. Then it turns out that:

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0} \qquad \text{if and only if} \qquad x\mathbf{a}' + y\mathbf{b}' + z\mathbf{c}' = \mathbf{0}$$

where x, y, and z are some real numbers.

This also holds for bigger or smaller collections of columns. Why? Essentially, since we are preforming **row** operations, the relationships between **columns** should be unaffected – we aren't "mixing" the columns together. For example:

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 2 & 1 & 3 \end{bmatrix} \underset{\sim}{R1 \leftrightarrow R2} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 2 & 1 & 3 \end{bmatrix} \underset{\sim}{2R1+R3} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \underset{\sim}{-R2+R3} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{-1 \times R1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice in the RREF (on the far right) we have that the final column is 2 times the first column plus -1 times the second column.

$$2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

So this must be true for all of the other matrices as well. In particular, we have that the third column of the original matrix is 2 times the first column plus -1 times the second column:

$$2\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

Why is this important? Well, first, Gauss-Jordan elimination typically requires a lot of computation. This correspondence gives you a way to check that you row-reduced correctly! We will see other applications of this correspondence later in the course.

Another example: Let A be a 3x3 matrix whose RREF is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Suppose that we know the first column of A is $\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$ and the second is $\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$. Then, by the linear correspondence, we know that the third column must be $-1 \cdot \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 2 \cdot \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$. Therefore, the mystery matrix is $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Homework Problems:

1. Verify that the linear correspondence holds between the each matrix, its REF, and its RREF in the previous homework problems and examples.

2. I just finished row reducing the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 5 & 0 \end{bmatrix}$ and got $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Something's wrong. Just using the linear correspondence explain how I know something's wrong and then find the real RREF (without row reducing).

3. A mystery matrix has the RREF of $\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The first column of this matrix is $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and the third column is $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. Find the mystery matrix.