Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$) such that $\dim(V) = n < \infty$. Let $T : V \to V$ be a linear transformation (since we are mapping from V to itself, we could refer to T as a linear operator).

Definition: Let $\mathbf{v} \in V$ such that $\mathbf{v} \neq \mathbf{0}$ and $T(\mathbf{v}) = \lambda\mathbf{v}$. Then $\mathbf{v}$ is an *eigenvector* for T with *eigenvalue* λ. Moreover, we say that $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) is an *eigenvalue* for T if T has an eigenvector with eigenvalue λ.

Note: While 0 can be an eigenvalue, $\mathbf{0}$ is not allowed to be an eigenvector. Otherwise, since $T(\mathbf{0}) = \mathbf{0} = \lambda\mathbf{0}$, we would have that every scalar is an eigenvalue of T and $\mathbf{0}$ would have every scalar as its eigenvalue!

Definition: Let $f(t) = \det(T - tI)$. Then $f(t)$ is called the *characteristic polynomial* of T.[1]

Note: λ is an eigenvalue of T $\Leftrightarrow$ there exists an non-zero vector $\mathbf{v}$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$ $\Leftrightarrow$ there exists a non-zero vector $\mathbf{v}$ such that $(T - \lambda I)(\mathbf{v}) = \mathbf{0}$ $\Leftrightarrow$ $\mathrm{Ker}(T - \lambda I) \neq \{\mathbf{0}\}$ $\Leftrightarrow$ $T - \lambda I$ is not 1-1 $\Leftrightarrow$ $T - \lambda I$ is not invertible $\Leftrightarrow$ $\det(T - \lambda I) \neq 0$. We have just proved...

Theorem: λ is an eigenvalue of T if and only if λ is a root of the characteristic polynomial of T (that is $f(\lambda) = \det(T - \lambda I) = 0$).

Facts: Let $f(t)$ be the characteristic polynomial of T. Then $f(t)$ is a polynomial of degree n whose leading coefficient is $(-1)^n$. In addition, $f(0) = \det(T)$ (the constant term is just the determinant of T). Also, the coefficient of t^{n-1} in $f(t)$ is $(-1)^{n-1}\mathrm{tr}(T)$ (i.e., $\pm$ the trace of T).

Definition: Factor T's characteristic polynomial (over $\mathbb{C}$): $f(t) = (-1)^n(t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k}$ (where $\lambda_i \neq \lambda_j$ for $i \neq j$ and $m_i > 0$). Then the roots of $f(t)$ (i.e. the eigenvalues of T) are $\lambda_1, \ldots, \lambda_k$. We say that the *algebraic multiplicity* of λ_i is m_i (the number of factors $(t - \lambda_i)$ appearing in the characteristic polynomial). Notice that the sum of the algebraic multiplicities is $n = \dim(V)$ (the degree of the characteristic polynomial).
Technical note: More accurately, if we are working over $\mathbb{R}$, the non-real roots are not actually eigenvalues.

Definition: Let $E_\lambda = \mathrm{Ker}(T - \lambda I) = \{\mathbf{v} \in V \mid (T - \lambda I)(\mathbf{v}) = \mathbf{0}\} = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \lambda\mathbf{v}\} = \{\mathbf{0}\} \cup \{\mathbf{v} \in V \mid \mathbf{v}$ is an eigenvector of T with eigenvalue $\lambda\}$. If $E_\lambda \neq \{\mathbf{0}\}$ (i.e., λ is an eigenvalue), then we call E_λ an *eigenspace* of T. Notice that E_λ is a subspace of V (since it is the kernel of a linear transformation).

Definition: $\dim(E_\lambda) = \dim(\mathrm{Ker}(T - \lambda I)) = \mathrm{nullity}(T - \lambda I)$ is called the *geometric multiplicity* of λ. This is the number of linearly independent eigenvectors with eigenvalue λ. Notice that if λ is an eigenvalue then E_λ cannot be the zero subspace. Thus geometric multiplicities of eigenvalues are always at least 1.

Theorem: Let λ be an eigenvalue of T with algebraic mult. m and geometric mult. g. Then $1 \leq g \leq m$.

Theorem: Eigenvectors with different eigenvalues are linearly independent. Moreover, if S_i is a linearly independent set of eigenvectors with eigenvalue λ_i and $\lambda_i \neq \lambda_j$ for $i \neq j$, then $S_1 \dot\cup S_2 \dot\cup \cdots \dot\cup S_k$ is a linearly independent set.

Definition: T is *diagonalizable* if there is a basis for V consisting of eigenvectors for T. Notice if β is such a basis, then $[T]_\beta$ is a diagonal matrix!

Corollary: T is diagonalizable (over $\mathbb{R}$) if and only if the eigenvalues of T all belong to $\mathbb{R}$ (i.e. the characteristic polynomial completely factors over $\mathbb{R}$) and the geometric and algebraic multiplicities of each eigenvalue match.

[1] This is the definition in Lay. Other texts define $g(t) = \det(tI - T)$ to be the characteristic polynomial. Notice that $f(t) = (-1)^n g(t)$ where $n = \dim(V)$. So for even sized matrices $f = g$ and for odd sized matrices $f = -g$.