Math 2240 # Homework #2 **Due: August 3rd, 2018**

Be sure to show your work!

1. Here we explore eigen-stuff. You definitely do not want to (and probably cannot) do these by hand.

For example:

```
[> restart;
[> with(LinearAlgebra):
[> A := <<5,-1,-2,-3>|<5,11,2,3>|<7,5,11,6>|<-2,-4,1,7>>;
[> Eigenvectors(A);
```

This will result in...

$$\begin{bmatrix} 10 \\ 10 \\ 7 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

This means that A has eigenvalues 10 and 7 both with algebraic multiplicity 2. The 4×4 matrix consists of eigenvectors (and the zero vector). The first two columns are (linearly independent) eigenvectors with eigenvalue 10 (i.e., the geometric multiplicity of $\lambda = 10$ is $\dim(E_{10}) = 2$). The third column is an eigenvector with eigenvalue 7. The column of 0's indicates that there isn't a second dimension's worth of eigenvectors with eigenvalue 7 (i.e., the geometric multiplicity of $\lambda = 7$ is $\dim(E_7) = 1$).

Functions of matrices: When a matrix A is diagonalizable we can compute "$f(A)$" for any function $f(x)$. Specifically, suppose that $P^{-1}AP = D$ where $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ is a diagonal matrix. Then $f(A) = Pf(D)P^{-1} = P \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} P^{-1}$.

Why do we define $f(A)$ this way? Notice that since $P^{-1}AP = D$, we have $A = PDP^{-1}$. Thus if $f(x) = \sum_{k=0}^{\infty} c_k x^k$, then $f(A) = \sum_{k=0}^{\infty} c_k A^k = \sum_{k=0}^{\infty} c_k (PDP^{-1})^k = \sum_{k=0}^{\infty} c_k PD^kP^{-1} = P\left(\sum_{k=0}^{\infty} c_k D^k\right)P^{-1} = Pf(D)P^{-1}$. When D is diagonal, $f(D)$ is just $f(x)$ applied to the diagonal entries of D.

For example: $A = \begin{bmatrix} 5 & 3 \\ -2 & 0 \end{bmatrix}$. Then if $P = \begin{bmatrix} -1 & -3 \\ 1 & 2 \end{bmatrix}$, then $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = D$.

This means that $e^A = Pe^DP^{-1} = P \begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix} P^{-1} = \begin{bmatrix} -2e^2 + 3e^3 & -3e^2 + 3e^3 \\ 2e^2 - 2e^3 & 3e^2 - 2e^3 \end{bmatrix}$.

Note: You may also find the "`simplify(stuff);`" command helpful. If you want decimal approximations use "`evalf(stuff);`".

(a) Let $A = \begin{bmatrix} 6 & -1 & 7 & 1 \\ 1 & -3 & 0 & 1 \\ -1 & 1 & -2 & -1 \\ -7 & 0 & -7 & -2 \end{bmatrix}$. Find the eigenvalues of A. Then find each eigenvalue's algebraic and geometric multiplicities and a basis for each eigenspace for A. Is A diagonalizable? Why or why not?

```
A := <<6,1,-1,-7>|<-1,-3,1,0>|<7,0,-2,-7>|<1,1,-1,-2>>;
```

(b) Let $B = \begin{bmatrix} 13 & -12 & 9 & -15 & 9 \\ 6 & -5 & 9 & -15 & 9 \\ 6 & -12 & -5 & 6 & 9 \\ 6 & -12 & 9 & -8 & 9 \\ -6 & 12 & 12 & -6 & -2 \end{bmatrix}$. Find a matrix P which diagonalizes B. What is $D = P^{-1}BP$ for your matrix P? Use this to compute $\sin(B)$ (the matrix sine of B). Please round your matrix entries (3 decimal places is fine).

```
B := <<13,6,6,6,-6>|<-12,-5,-12,-12,12>|<9,9,-5,9,12>|<-15,-15,6,-8,-6>|<9,9,9,9,-2>>;
```

(c) Let $C = \begin{bmatrix} 4 & 5 & 0 & -5 \\ -5 & 24 & -5 & -20 \\ 5 & -5 & 9 & 5 \\ -5 & 15 & -5 & -11 \end{bmatrix}$. Again, find a matrix P which diagonalizes C. What is $D = P^{-1}CP$ for your matrix P? Use this to compute $\sqrt{C}$. Then check that $(\sqrt{C})^2 = C$.

```
C := <<4,-5,5,-5>|<5,24,-5,15>|<0,-5,9,-5>|<-5,-20,5,-11>>;
```

2. Let $W = \mathrm{Span}(S) \subseteq \mathbb{P}_3$ where $S = \{t^3 + 2t - 1, -2t^3 - 4t + 2, t^3 + 2t^2 + 3t + 1, t^3 - 4t^2 - 5\}$.

(a) Find a basis α for W.

(b) Decide if any of $f(t) = t^3 + t$, $g(t) = t^3 - 2t^2 + t - 3$, or $h(t) = 2t^2 + 3$ belong to W. For those that belong to W, find their α-coordinate vectors.

(c) Extend α to a basis β for $\mathbb{P}_3$.

(d) Suppose that $[p(t)]_\beta = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. What is $p(t)$?

3. Let $\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ and $\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$.

(a) Show that both α and β are bases for $\mathbb{R}^{2\times 2} = M_{2\times 2}$.

(b) Find the change of coordinate matrix $P = \mathcal{P}_{\beta\leftarrow\alpha} = [I]_\alpha^\beta$ (changing from α- to β-coordinates).

(c) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find $[A]_\alpha$. Then use part (b) to compute $[A]_\beta$.

4. Let $W = \{f(t) \in \mathbb{P}_3 \mid f''(1) = 0 \text{ and } f(-1) = 0\}$.

(a) Consider $T : \mathbb{P}_3 \to \mathbb{R}^2$ defined by $T(f(t)) = \begin{bmatrix} f''(1) \\ f(-1) \end{bmatrix}$. Show that T is a linear map. Then briefly explain why $W = \mathrm{Ker}(T)$.
Note: Since we now know W is a kernel (i.e., nullspace) of a linear transformation and kernels are subspaces, we have that W is a subspace of $\mathbb{P}_3$.

(b) Let $\alpha = \{1, t, t^2, t^3\}$ (the standard basis for $\mathbb{P}_3$). Let $\beta = \{\mathbf{e}_1, \mathbf{e}_2\}$ (the standard basis for $\mathbb{R}^2$). Find the coordinate matrix $[T]_\alpha^\beta$. Find the rank and nullity of T. Is T one-to-one? Onto?

(c) Find a basis for the null space of $[T]_\alpha^\beta$. Use this to get a basis γ for $W = \mathrm{Ker}(T)$.

(d) Show that $f(t) = t^3 - 3t^2 + 5t + 9 \in W$. Find $[f(t)]_\gamma$.

(e) Suppose that $[g(t)]_\gamma = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. What is $g(t)$?