Math 2240 # Homework #3 **Due: Tues., July 28^{st}, 2020**

Be sure to show your work!

Notes:

- This first homework was shorter. These later homeworks are longer. I don't plan on weighting each homework equally.
- You can copy/paste my sample code into Maple. However, if you would prefer to do computations using different software, that is ok. Make sure you turn in *all* of your work (including computer code).
- If you end up doing the entire assignment in Maple, feel free to turn in a Maple worksheet (.mw file) instead of a pdf.
- **While some calculations can be done by hand**, some are better done using technology. A calculator is a poor choice since I want *exact* answers not decimal approximations.

1. Find the determinant and inverse of the following 4×4 matrix (a, b, c, d are unknown constants):

$$X = \begin{bmatrix} a & b & c & d \\ b & 0 & 0 & c \\ c & 0 & 0 & b \\ d & c & b & a \end{bmatrix}$$

2. Let $W = \{f(t) \in \mathbb{P}_3 \mid f''(1) = 0 \text{ and } f(-1) = 0\}$.

(a) Consider $T : \mathbb{P}_3 \to \mathbb{R}^2$ defined by $T(f(t)) = \begin{bmatrix} f''(1) \\ f(-1) \end{bmatrix}$. Show that T is a linear map. Then briefly explain why $W = \text{Ker}(T)$.

Note: Since we now know W is a kernel (i.e., nullspace) of a linear transformation and kernels are subspaces, we have that W is a subspace of $\mathbb{P}_3$.

(b) Let $\alpha = \{1, t, t^2, t^3\}$ (the standard basis for $\mathbb{P}_3$). Let $\beta = \{\mathbf{e}_1, \mathbf{e}_2\}$ (the standard basis for $\mathbb{R}^2$). Find the coordinate matrix $[T]_\alpha^\beta$. Find the rank and nullity of T. Is T one-to-one? Onto?

(c) Find a basis for the null space of $[T]_\alpha^\beta$. Use this to get a basis γ for $W = \text{Ker}(T)$.

(d) Show that $f(t) = t^3 - 3t^2 + 5t + 9 \in W$. Find $[f(t)]_\gamma$.

(e) Suppose that $[g(t)]_\gamma = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. What is $g(t)$?

[Problem #3 is on the next page.]

3. Here we explore eigen-stuff. You definitely do not want to (and probably cannot) do these by hand.

For example:

```
restart;
with(LinearAlgebra):
A := <<5,-1,-2,-3>|<5,11,2,3>|<7,5,11,6>|<-2,-4,1,7>>;
Eigenvectors(A);
```

This will result in...

$$\begin{bmatrix}10\\10\\7\\7\end{bmatrix}, \quad \begin{bmatrix}1&1&1&0\\0&1&-1&0\\1&0&1&0\\1&0&0&0\end{bmatrix}$$

This means that A has eigenvalues 10 and 7 both with algebraic multiplicity 2. The 4×4 matrix consists of eigenvectors (and the zero vector). The first two columns are (linearly independent) eigenvectors with eigenvalue 10 (i.e., the geometric multiplicity of $\lambda = 10$ is $\dim(E_{10}) = 2$). The third column is an eigenvector with eigenvalue 7. The column of 0's indicates that there isn't a second dimension's worth of eigenvectors with eigenvalue 7 (i.e., the geometric multiplicity of $\lambda = 7$ is $\dim(E_7) = 1$).

Functions of matrices: When a matrix A is diagonalizable we can compute "$f(A)$" for any function $f(x)$. Specifically, suppose that $P^{-1}AP = D$ where $D = \begin{bmatrix}\lambda_1 && \\ & \ddots & \\ && \lambda_n\end{bmatrix}$ is a diagonal matrix. Then $f(A) = Pf(D)P^{-1} = P\begin{bmatrix}f(\lambda_1) && \\ & \ddots & \\ && f(\lambda_n)\end{bmatrix}P^{-1}$.

Why do we define $f(A)$ this way? Notice that since $P^{-1}AP = D$, we have $A = PDP^{-1}$. Thus if $f(x) = \sum_{k=0}^{\infty} c_k x^k$, then $f(A) = \sum_{k=0}^{\infty} c_k A^k = \sum_{k=0}^{\infty} c_k (PDP^{-1})^k = \sum_{k=0}^{\infty} c_k PD^kP^{-1} = P\left(\sum_{k=0}^{\infty} c_k D^k\right)P^{-1} = Pf(D)P^{-1}$. When D is diagonal, $f(D)$ is just $f(x)$ applied to the diagonal entries of D.

For example: $A = \begin{bmatrix}5&3\\-2&0\end{bmatrix}$. Then if $P = \begin{bmatrix}-1&-3\\1&2\end{bmatrix}$, then $P^{-1}AP = \begin{bmatrix}2&0\\0&3\end{bmatrix} = D$.

This means that $e^A = Pe^DP^{-1} = P\begin{bmatrix}e^2&0\\0&e^3\end{bmatrix}P^{-1} = \begin{bmatrix}-2e^2+3e^3 & -3e^2+3e^3\\ 2e^2-2e^3 & 3e^2-2e^3\end{bmatrix}$.

Note: You may also find the "`simplify(stuff);`" command helpful. If you want decimal approximations use "`evalf(stuff);`".

(a) Let $A = \begin{bmatrix}6&-1&7&1\\1&-3&0&1\\-1&1&-2&-1\\-7&0&-7&-2\end{bmatrix}$. Find the eigenvalues of A. Then find each eigenvalue's algebraic and geometric multiplicities and a basis for each eigenspace for A. Is A diagonalizable? Why or why not?

```
A := <<6,1,-1,-7>|<-1,-3,1,0>|<7,0,-2,-7>|<1,1,-1,-2>>;
```

(b) Let $B = \begin{bmatrix}13&-12&9&-15&9\\6&-5&9&-15&9\\6&-12&-5&6&9\\6&-12&9&-8&9\\-6&12&12&-6&-2\end{bmatrix}$. Find a matrix P which diagonalizes B. What is $D = P^{-1}BP$ for your matrix P? Use this to compute $\sin(B)$ (the matrix sine of B). Please round your matrix entries (3 decimal places is fine).

```
B := <<13,6,6,6,-6>|<-12,-5,-12,-12,12>|<9,9,-5,9,12>|<-15,-15,6,-8,-6>|<9,9,9,9,-2>>;
```

(c) Let $C = \begin{bmatrix}4&5&0&-5\\-5&24&-5&-20\\5&-5&9&5\\-5&15&-5&-11\end{bmatrix}$. Again, find a matrix P which diagonalizes C. What is $D = P^{-1}CP$ for your matrix P? Use this to compute $\sqrt{C}$. Then check that $(\sqrt{C})^2 = C$.

```
C := <<4,-5,5,-5>|<5,24,-5,15>|<0,-5,9,-5>|<-5,-20,5,-11>>;
```