28. Show that the space $C(\mathbb{R})$ of all continuous functions defined on the real line is an infinite-dimensional space.

In Exercises 29 and 30, V is a nonzero finite-dimensional vector space, and the vectors listed belong to V. Mark each statement True or False. Justify each answer. (These questions are more difficult than those in Exercises 19 and 20.)

29. a. If there exists a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ that spans V, then $\dim V \le p$.

b. If there exists a linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then $\dim V \ge p$.

c. If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.

30. a. If there exists a linearly dependent set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in V, then $\dim V \le p$.

b. If every set of p elements in V fails to span V, then $\dim V > p$.

c. If $p \ge 2$ and $\dim V = p$, then every set of $p - 1$ nonzero vectors is linearly independent.

Exercises 31 and 32 concern finite-dimensional vector spaces V and W and a linear transformation $T : V \to W$.

31. Let H be a nonzero subspace of V, and let $T(H)$ be the set of images of vectors in H. Then $T(H)$ is a subspace of W, by Exercise 35 in Section 4.2. Prove that $\dim T(H) \le \dim H$.

32. Let H be a nonzero subspace of V, and suppose T is a one-to-one (linear) mapping of V into W. Prove that $\dim T(H) = \dim H$. If T happens to be a one-to-one mapping of V *onto* W, then $\dim V = \dim W$. Isomorphic finite-dimensional vector spaces have the same dimension.

33. **[M]** According to Theorem 11, a linearly independent set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ can be expanded to a basis for $\mathbb{R}^n$. One way to do this is to create $A = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_k \ \mathbf{e}_1 \ \cdots \ \mathbf{e}_n\,]$, with $\mathbf{e}_1, \dots, \mathbf{e}_n$ the columns of the identity matrix; the pivot columns of A form a basis for $\mathbb{R}^n$.

a. Use the method described to extend the following vectors to a basis for $\mathbb{R}^5$:

$$\mathbf{v}_1 = \begin{bmatrix} -9 \\ -7 \\ 8 \\ -5 \\ 7 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 6 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 7 \\ -8 \\ 5 \\ -7 \end{bmatrix}$$

b. Explain why the method works in general: Why are the original vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ included in the basis found for Col A? Why is Col $A = \mathbb{R}^n$?

34. **[M]** Let $\mathcal{B} = \{1, \cos t, \cos^2 t, \dots, \cos^6 t\}$ and $\mathcal{C} = \{1, \cos t, \cos 2t, \dots, \cos 6t\}$. Assume the following trigonometric identities (see Exercise 37 in Section 4.1).

$$\begin{aligned} \cos 2t &= -1 + 2\cos^2 t \\ \cos 3t &= -3\cos t + 4\cos^3 t \\ \cos 4t &= 1 - 8\cos^2 t + 8\cos^4 t \\ \cos 5t &= 5\cos t - 20\cos^3 t + 16\cos^5 t \\ \cos 6t &= -1 + 18\cos^2 t - 48\cos^4 t + 32\cos^6 t \end{aligned}$$

Let H be the subspace of functions spanned by the functions in $\mathcal{B}$. Then $\mathcal{B}$ is a basis for H, by Exercise 38 in Section 4.3.

a. Write the $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$, and use them to show that $\mathcal{C}$ is a linearly independent set in H.

b. Explain why $\mathcal{C}$ is a basis for H.

SOLUTIONS TO PRACTICE PROBLEMS

1. False. Consider the set $\{\mathbf{0}\}$.

2. True. By the Spanning Set Theorem, S contains a basis for V; call that basis S'. Then T will contain more vectors than S'. By Theorem 9, T is linearly dependent.

4.6 RANK

With the aid of vector space concepts, this section takes a look *inside* a matrix and reveals several interesting and useful relationships hidden in its rows and columns.

For instance, imagine placing 2000 random numbers into a 40×50 matrix A and then determining both the maximum number of linearly independent columns in A and the maximum number of linearly independent columns in A^T (rows in A). Remarkably, the two numbers are the same. As we'll soon see, their common value is the *rank* of the matrix. To explain why, we need to examine the subspace spanned by the rows of A.

The Row Space

If A is an $m \times n$ matrix, each row of A has n entries and thus can be identified with a vector in $\mathbb{R}^n$. The set of all linear combinations of the row vectors is called the **row space** of A and is denoted by Row A. Each row has n entries, so Row A is a subspace of $\mathbb{R}^n$. Since the rows of A are identified with the columns of A^T, we could also write Col A^T in place of Row A.

EXAMPLE 1 Let

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \text{and} \quad \begin{aligned} \mathbf{r}_1 &= (-2, -5, 8, 0, -17) \\ \mathbf{r}_2 &= (1, 3, -5, 1, 5) \\ \mathbf{r}_3 &= (3, 11, -19, 7, 1) \\ \mathbf{r}_4 &= (1, 7, -13, 5, -3) \end{aligned}$$

The row space of A is the subspace of $\mathbb{R}^5$ spanned by $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$. That is, Row A = Span $\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$. It is natural to write row vectors horizontally; however, they may also be written as column vectors if that is more convenient. ■

If we knew some linear dependence relations among the rows of matrix A in Example 1, we could use the Spanning Set Theorem to shrink the spanning set to a basis. Unfortunately, row operations on A will not give us that information, because row operations change the row-dependence relations. But row reducing A is certainly worthwhile, as the next theorem shows!

THEOREM 13

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

PROOF If B is obtained from A by row operations, the rows of B are linear combinations of the rows of A. It follows that any linear combination of the rows of B is automatically a linear combination of the rows of A. Thus the row space of B is contained in the row space of A. Since row operations are reversible, the same argument shows that the row space of A is a subset of the row space of B. So the two row spaces are the same. If B is in echelon form, its nonzero rows are linearly independent because no nonzero row is a linear combination of the nonzero rows below it. (Apply Theorem 4 to the nonzero rows of B in reverse order, with the first row last.) Thus the nonzero rows of B form a basis of the (common) row space of B and A. ■

The main result of this section involves the three spaces: Row A, Col A, and Nul A. The following example prepares the way for this result and shows how *one* sequence of row operations on A leads to bases for all three spaces.

EXAMPLE 2 Find bases for the row space, the column space, and the null space of the matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

SOLUTION To find bases for the row space and the column space, row reduce A to an echelon form:

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 13, the first three rows of B form a basis for the row space of A (as well as for the row space of B). Thus

$$\text{Basis for Row } A\text{: } \{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$$

For the column space, observe from B that the pivots are in columns 1, 2, and 4. Hence columns 1, 2, and 4 of A (not B) form a basis for Col A:

$$\text{Basis for Col } A\text{: } \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

Notice that any echelon form of A provides (in its nonzero rows) a basis for Row A and also identifies the pivot columns of A for Col A. However, for Nul A, we need the *reduced echelon form*. Further row operations on B yield

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{0}$ is equivalent to $C\mathbf{x} = \mathbf{0}$, that is,

$$\begin{aligned} x_1 + \quad\quad x_3 \quad\quad + \ \ x_5 &= 0 \\ x_2 - 2x_3 \quad\quad + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

So $x_1 = -x_3 - x_5$, $x_2 = 2x_3 - 3x_5$, $x_4 = 5x_5$, with x_3 and x_5 free variables. The usual calculations (discussed in Section 4.2) show that

$$\text{Basis for Nul } A\text{: } \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Observe that, unlike the basis for Col A, the bases for Row A and Nul A have no simple connection with the entries in A itself.[1] ■

[1] It is possible to find a basis for the row space Row A that uses rows of A. First form A^T, and then row reduce until the pivot columns of A^T are found. These pivot columns of A^T are rows of A, and they form a basis for the row space of A.

Warning: Although the first three rows of B in Example 2 are linearly independent, it is wrong to conclude that the first three rows of A are linearly independent. (In fact, the third row of A is 2 times the first row plus 7 times the second row.) Row operations may change the linear dependence relations among the *rows* of a matrix.

The Rank Theorem

WEB

The next theorem describes fundamental relations among the dimensions of Col A, Row A, and Nul A.

DEFINITION

The **rank** of A is the dimension of the column space of A.

Since Row A is the same as Col A^T, the dimension of the row space of A is the rank of A^T. The dimension of the null space is sometimes called the **nullity** of A, though we will not use this term.

An alert reader may have already discovered part or all of the next theorem while working the exercises in Section 4.5 or reading Example 2 above.

THEOREM 14

The Rank Theorem

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n$$

PROOF By Theorem 6 in Section 4.3, rank A is the number of pivot columns in A. Equivalently, rank A is the number of pivot positions in an echelon form B of A. Furthermore, since B has a nonzero row for each pivot, and since these rows form a basis for the row space of A, the rank of A is also the dimension of the row space.

From Section 4.5, the dimension of Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$. Expressed another way, the dimension of Nul A is the number of columns of A that are *not* pivot columns. (It is the number of these columns, not the columns themselves, that is related to Nul A.) Obviously,

$$\begin{Bmatrix} \text{number of} \\ \text{pivot columns} \end{Bmatrix} + \begin{Bmatrix} \text{number of} \\ \text{nonpivot columns} \end{Bmatrix} = \begin{Bmatrix} \text{number of} \\ \text{columns} \end{Bmatrix}$$

This proves the theorem. ■

The ideas behind Theorem 14 are visible in the calculations in Example 2. The three pivot positions in the echelon form B determine the basic variables and identify the basis vectors for Col A and those for Row A.

EXAMPLE 3

a. If A is a 7×9 matrix with a two-dimensional null space, what is the rank of A?

b. Could a 6×9 matrix have a two-dimensional null space?

SOLUTION

a. Since A has 9 columns, $(\text{rank } A) + 2 = 9$, and hence $\text{rank } A = 7$.

b. No. If a 6×9 matrix, call it B, had a two-dimensional null space, it would have to have rank 7, by the Rank Theorem. But the columns of B are vectors in $\mathbb{R}^6$, and so the dimension of Col B cannot exceed 6; that is, rank B cannot exceed 6. ∎

The next example provides a nice way to visualize the subspaces we have been studying. In Chapter 6, we will learn that Row A and Nul A have only the zero vector in common and are actually "perpendicular" to each other. The same fact will apply to Row A^T (= Col A) and Nul A^T. So Fig. 1, which accompanies Example 4, creates a good mental image for the general case. (The value of studying A^T along with A is demonstrated in Exercise 29.)

EXAMPLE 4 Let $A = \begin{bmatrix} 3 & 0 & -1 \\ 3 & 0 & -1 \\ 4 & 0 & 5 \end{bmatrix}$. It is readily checked that Nul A is the x_2-axis, Row A is the x_1x_3-plane, Col A is the plane whose equation is $x_1 - x_2 = 0$, and Nul A^T is the set of all multiples of $(1, -1, 0)$. Figure 1 shows Nul A and Row A in the domain of the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$; the range of this mapping, Col A, is shown in a separate copy of $\mathbb{R}^3$, along with Nul A^T. ∎

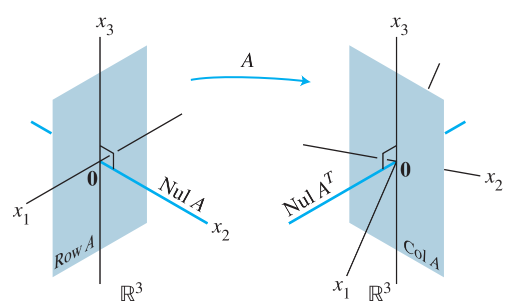

FIGURE 1 Subspaces determined by a matrix A.

Applications to Systems of Equations

The Rank Theorem is a powerful tool for processing information about systems of linear equations. The next example simulates the way a real-life problem using linear equations might be stated, without explicit mention of linear algebra terms such as matrix, subspace, and dimension.

EXAMPLE 5 A scientist has found two solutions to a homogeneous system of 40 equations in 42 variables. The two solutions are not multiples, and all other solutions can be constructed by adding together appropriate multiples of these two solutions. Can the scientist be *certain* that an associated nonhomogeneous system (with the same coefficients) has a solution?

SOLUTION Yes. Let A be the 40×42 coefficient matrix of the system. The given information implies that the two solutions are linearly independent and span Nul A. So $\dim \text{Nul } A = 2$. By the Rank Theorem, $\dim \text{Col } A = 42 - 2 = 40$. Since $\mathbb{R}^{40}$ is the only subspace of $\mathbb{R}^{40}$ whose dimension is 40, Col A must be all of $\mathbb{R}^{40}$. This means that every nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ has a solution. ∎

Rank and the Invertible Matrix Theorem

The various vector space concepts associated with a matrix provide several more statements for the Invertible Matrix Theorem. The new statements listed here follow those in the original Invertible Matrix Theorem in Section 2.3.

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$.
n. $\text{Col}\, A = \mathbb{R}^n$
o. $\dim \text{Col}\, A = n$
p. $\text{rank}\, A = n$
q. $\text{Nul}\, A = \{\mathbf{0}\}$
r. $\dim \text{Nul}\, A = 0$

PROOF Statement (m) is logically equivalent to statements (e) and (h) regarding linear independence and spanning. The other five statements are linked to the earlier ones of the theorem by the following chain of almost trivial implications:

$$(\text{g}) \Rightarrow (\text{n}) \Rightarrow (\text{o}) \Rightarrow (\text{p}) \Rightarrow (\text{r}) \Rightarrow (\text{q}) \Rightarrow (\text{d})$$

Statement (g), which says that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$, implies (n), because Col A is precisely the set of all $\mathbf{b}$ such that the equation $A\mathbf{x} = \mathbf{b}$ is consistent. The implications (n) $\Rightarrow$ (o) $\Rightarrow$ (p) follow from the definitions of dimension and rank. If the rank of A is n, the number of columns of A, then $\dim \text{Nul}\, A = 0$, by the Rank Theorem, and so $\text{Nul}\, A = \{\mathbf{0}\}$. Thus (p) $\Rightarrow$ (r) $\Rightarrow$ (q). Also, (q) implies that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is statement (d). Since statements (d) and (g) are already known to be equivalent to the statement that A is invertible, the proof is complete. ■

SG Expanded Table for the IMT 4–19

We have refrained from adding to the Invertible Matrix Theorem obvious statements about the row space of A, because the row space is the column space of A^T. Recall from statement (l) of the Invertible Matrix Theorem that A is invertible if and only if A^T is invertible. Hence every statement in the Invertible Matrix Theorem can also be stated for A^T. To do so would double the length of the theorem and produce a list of over 30 statements!

NUMERICAL NOTE

Many algorithms discussed in this text are useful for understanding concepts and making simple computations by hand. However, the algorithms are often unsuitable for large-scale problems in real life.

Rank determination is a good example. It would seem easy to reduce a matrix to echelon form and count the pivots. But unless exact arithmetic is performed on a matrix whose entries are specified exactly, row operations can change the apparent rank of a matrix. For instance, if the value of x in the matrix $\begin{bmatrix} 5 & 7 \\ 5 & x \end{bmatrix}$ is not stored exactly as 7 in a computer, then the rank may be 1 or 2, depending on whether the computer treats $x - 7$ as zero.

In practical applications, the effective rank of a matrix A is often determined from the singular value decomposition of A, to be discussed in Section 7.4. This decomposition is also a reliable source of bases for Col A, Row A, Nul A, and Nul A^T.

WEB

PRACTICE PROBLEMS

The matrices below are row equivalent.

$$A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -4 & 3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1. Find rank A and dim Nul A.
2. Find bases for Col A and Row A.
3. What is the next step to perform to find a basis for Nul A?
4. How many pivot columns are in a row echelon form of A^T?

4.6 EXERCISES

In Exercises 1–4, assume that the matrix A is row equivalent to B. Without calculations, list rank A and dim Nul A. Then find bases for Col A, Row A, and Nul A.

1. $$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. $$A = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 2 & 6 & 6 & 0 & -3 \\ 3 & 9 & 3 & 6 & -3 \\ 3 & 9 & 0 & 9 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 4 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. $$A = \begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ -2 & -3 & 6 & -3 & 0 & -6 \\ 4 & 9 & -12 & 9 & 3 & 12 \\ -2 & 3 & 6 & 3 & 3 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 6 & -6 & 6 & 3 & 6 \\ 0 & 3 & 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4. $$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5. If a 4×7 matrix A has rank 3, find dim Nul A, dim Row A, and rank A^T.

6. If a 7×5 matrix A has rank 2, find dim Nul A, dim Row A, and rank A^T.

7. Suppose a 4×7 matrix A has four pivot columns. Is Col $A = \mathbb{R}^4$? Is Nul $A = \mathbb{R}^3$? Explain your answers.

8. Suppose a 6×8 matrix A has four pivot columns. What is dim Nul A? Is Col $A = \mathbb{R}^4$? Why or why not?

9. If the null space of a 4×6 matrix A is 3-dimensional, what is the dimension of the column space of A? Is Col $A = \mathbb{R}^3$? Why or why not?

10. If the null space of an 8×7 matrix A is 5-dimensional, what is the dimension of the column space of A?

11. If the null space of an 8×5 matrix A is 3-dimensional, what is the dimension of the row space of A?

12. If the null space of a 5×4 matrix A is 2-dimensional, what is the dimension of the row space of A?

13. If A is a 7×5 matrix, what is the largest possible rank of A? If A is a 5×7 matrix, what is the largest possible rank of A? Explain your answers.

14. If A is a 5×4 matrix, what is the largest possible dimension of the row space of A? If A is a 4×5 matrix, what is the largest possible dimension of the row space of A? Explain.

15. If A is a 3×7 matrix, what is the smallest possible dimension of Nul A?

16. If A is a 7×5 matrix, what is the smallest possible dimension of Nul A?

In Exercises 17 and 18, A is an $m \times n$ matrix. Mark each statement True or False. Justify each answer.

17. a. The row space of A is the same as the column space of A^T.

b. If B is any echelon form of A, and if B has three nonzero rows, then the first three rows of A form a basis for Row A.

c. The dimensions of the row space and the column space of A are the same, even if A is not square.

d. The sum of the dimensions of the row space and the null space of A equals the number of rows in A.

e. On a computer, row operations can change the apparent rank of a matrix.

18. a. If B is any echelon form of A, then the pivot columns of B form a basis for the column space of A.

b. Row operations preserve the linear dependence relations among the rows of A.

c. The dimension of the null space of A is the number of columns of A that are *not* pivot columns.

d. The row space of A^T is the same as the column space of A.

e. If A and B are row equivalent, then their row spaces are the same.

19. Suppose the solutions of a homogeneous system of five linear equations in six unknowns are all multiples of one nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations? Explain.

20. Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution, with two free variables. Is it possible to change some constants on the equations' right sides to make the new system inconsistent? Explain.

21. Suppose a nonhomogeneous system of nine linear equations in ten unknowns has a solution for all possible constants on the right sides of the equations. Is it possible to find two nonzero solutions of the associated homogeneous system that are *not* multiples of each other? Discuss.

22. Is is possible that all solutions of a homogeneous system of ten linear equations in twelve variables are multiples of one fixed nonzero solution? Discuss.

23. A homogeneous system of twelve linear equations in eight unknowns has two fixed solutions that are not multiples of each other, and all other solutions are linear combinations of these two solutions. Can the set of all solutions be described with fewer than twelve homogeneous linear equations? If so, how many? Discuss.

24. Is it possible for a nonhomogeneous system of seven equations in six unknowns to have a unique solution for some right-hand side of constants? Is it possible for such a system to have a unique solution for every right-hand side? Explain.

25. A scientist solves a nonhomogeneous system of ten linear equations in twelve unknowns and finds that three of the unknowns are free variables. Can the scientist be certain that, if the right sides of the equations are changed, the new nonhomogeneous system will have a solution? Discuss.

26. In statistical theory, a common requirement is that a matrix be of *full rank*. That is, the rank should be as large as possible. Explain why an $m \times n$ matrix with more rows than columns has full rank if and only if its columns are linearly independent.

Exercises 27–29 concern an $m \times n$ matrix A and what are often called the *fundamental subspaces* determined by A.

27. Which of the subspaces Row A, Col A, Nul A, Row A^T, Col A^T, and Nul A^T are in $\mathbb{R}^m$ and which are in $\mathbb{R}^n$? How many distinct subspaces are in this list?

28. Justify the following equalities:

a. $\dim \text{Row } A + \dim \text{Nul } A = n$ Number of columns of A

b. $\dim \text{Col } A + \dim \text{Nul } A^T = m$ Number of rows of A

29. Use Exercise 28 to explain why the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^m$ if and only if the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

30. Suppose A is $m \times n$ and $\mathbf{b}$ is in $\mathbb{R}^m$. What has to be true about the two numbers rank $[\,A \quad \mathbf{b}\,]$ and rank A in order for the equation $A\mathbf{x} = \mathbf{b}$ to be consistent?

Rank 1 matrices are important in some computer algorithms and several theoretical contexts, including the singular value decomposition in Chapter 7. It can be shown that an $m \times n$ matrix A has rank 1 if and only if it is an outer product; that is, $A = \mathbf{u}\mathbf{v}^T$ for some $\mathbf{u}$ in $\mathbb{R}^m$ and $\mathbf{v}$ in $\mathbb{R}^n$. Exercises 31–33 suggest why this property is true.

31. Verify that rank $\mathbf{u}\mathbf{v}^T \le 1$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

32. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $\mathbf{v}$ in $\mathbb{R}^3$ such that $\begin{bmatrix} 1 & -3 & 4 \\ 2 & -6 & 8 \end{bmatrix} = \mathbf{u}\mathbf{v}^T$.

33. Let A be any 2×3 matrix such that rank $A = 1$, let $\mathbf{u}$ be the first column of A, and suppose $\mathbf{u} \ne \mathbf{0}$. Explain why there is a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $A = \mathbf{u}\mathbf{v}^T$. How could this construction be modified if the first column of A were zero?

34. Let A be an $m \times n$ matrix of rank $r > 0$ and let U be an echelon form of A. Explain why there exists an invertible matrix E such that $A = EU$, and use this factorization to write A as the sum of r rank 1 matrices. [*Hint:* See Theorem 10 in Section 2.4.]

35. [**M**] Let $A = \begin{bmatrix} 7 & -9 & -4 & 5 & 3 & -3 & -7 \\ -4 & 6 & 7 & -2 & -6 & -5 & 5 \\ 5 & -7 & -6 & 5 & -6 & 2 & 8 \\ -3 & 5 & 8 & -1 & -7 & -4 & 8 \\ 6 & -8 & -5 & 4 & 4 & 9 & 3 \end{bmatrix}$.

a. Construct matrices C and N whose columns are bases for Col A and Nul A, respectively, and construct a matrix R whose *rows* form a basis for Row A.

b. Construct a matrix M whose columns form a basis for Nul A^T, form the matrices $S = [\,R^T \quad N\,]$ and $T = [\,C \quad M\,]$, and explain why S and T should be square. Verify that both S and T are invertible.

36. [**M**] Repeat Exercise 35 for a random integer-valued 6×7 matrix A whose rank is at most 4. One way to make A is to create a random integer-valued 6×4 matrix J and a random integer-valued 4×7 matrix K, and set $A = JK$. (See Supplementary Exercise 12 at the end of the chapter; and see the *Study Guide* for matrix-generating programs.)

37. [**M**] Let A be the matrix in Exercise 35. Construct a matrix C whose columns are the pivot columns of A, and construct a matrix R whose rows are the nonzero rows of the reduced echelon form of A. Compute CR, and discuss what you see.

38. [**M**] Repeat Exercise 37 for three random integer-valued 5×7 matrices A whose ranks are 5, 4, and 3. Make a conjecture about how CR is related to A for any matrix A. Prove your conjecture.

SOLUTIONS TO PRACTICE PROBLEMS

1. A has two pivot columns, so rank $A = 2$. Since A has 5 columns altogether, dim Nul $A = 5 - 2 = 3$.

2. The pivot columns of A are the first two columns. So a basis for Col A is

$$\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 8 \\ -5 \end{bmatrix} \right\}$$

The nonzero rows of B form a basis for Row A, namely, $\{(1, -2, -4, 3, -2), (0, 3, 9, -12, 12)\}$. In this particular example, it happens that any two rows of A form a basis for the row space, because the row space is two-dimensional and none of the rows of A is a multiple of another row. In general, the nonzero rows of an echelon form of A should be used as a basis for Row A, not the rows of A itself.

3. For Nul A, the next step is to perform row operations on B to obtain the reduced echelon form of A.

SG Major Review of Key Concepts 4–22

4. Rank A^T = rank A, by the Rank Theorem, because Col A^T = Row A. So A^T has two pivot positions.

27. *Hint:* Use the fact that each $\mathbb{P}_n$ is a subspace of $\mathbb{P}$.

29. Justify each answer.

a. True **b.** True **c.** True

31. *Hint:* Since H is a nonzero subspace of a finite-dimensional space, H is finite-dimensional and has a basis, say, $\mathbf{v}_1, \dots, \mathbf{v}_p$. First show that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ spans $T(H)$.

33. [**M**] **a.** One basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_2, \mathbf{e}_3\}$. In fact, any two of the vectors $\mathbf{e}_2, \dots, \mathbf{e}_5$ will extend $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to a basis of $\mathbb{R}^5$.

Section 4.6, page 236

1. rank $A = 2$; $\dim \text{Nul}\, A = 2$;

Basis for Col A: $\begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$

Basis for Row A: $(1, 0, -1, 5), (0, -2, 5, -6)$

Basis for Nul A: $\begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

3. rank $A = 3$; $\dim \text{Nul}\, A = 3$;

Basis for Col A: $\begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 3 \\ 3 \end{bmatrix}$

Basis for Row A: $(2, 6, -6, 6, 3, 6), (0, 3, 0, 3, 3, 0), (0, 0, 0, 0, 3, 0)$

Basis for Nul A: $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

5. 4, 3, 3

7. Yes; no. Since Col A is a four-dimensional subspace of $\mathbb{R}^4$, it coincides with $\mathbb{R}^4$. The null space cannot be $\mathbb{R}^3$, because the vectors in Nul A have 7 entries. Nul A is a three-dimensional subspace of $\mathbb{R}^7$, by the Rank Theorem.

9. 3, no. Notice that the columns of a 4×6 matrix are in $\mathbb{R}^4$, rather than $\mathbb{R}^3$. Col A is a three-dimensional subspace of $\mathbb{R}^4$.

11. 2

13. 5, 5. In both cases, the number of pivots cannot exceed the number of columns or the number of rows.

15. 4 **17.** See the *Study Guide*.

19. Yes. Try to write an explanation before you consult the *Study Guide*.

21. No. Explain why.

23. Yes. Only six homogeneous linear equations are necessary.

25. No. Explain why.

27. Row A and Nul A are in $\mathbb{R}^n$; Col A and Nul A^T are in $\mathbb{R}^m$. There are only four distinct subspaces because Row A^T = Col A and Col A^T = Row A.

29. Recall that $\dim \text{Col}\, A = m$ precisely when Col $A = \mathbb{R}^m$, or equivalently, when the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$. By Exercise 28(b), $\dim \text{Col}\, A = m$ precisely when $\dim \text{Nul}\, A^T = 0$, or equivalently, when the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

31. $\mathbf{u}\mathbf{v}^T = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$. The columns are all multiples of $\mathbf{u}$, so Col $\mathbf{u}\mathbf{v}^T$ is one-dimensional, unless $a = b = c = 0$.

33. *Hint:* Let $A = [\,\mathbf{u} \;\; \mathbf{u}_2 \;\; \mathbf{u}_3\,]$. If $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{u}$ is a basis for Col A. Why?

35. [**M**] *Hint:* See Exercise 28 and the remarks before Example 4.

37. [**M**] The matrices C and R given for Exercise 35 work here, and $A = CR$.

Section 4.7, page 242

1. **a.** $\begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$ **b.** $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$

3. (ii)

5. **a.** $\begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$

7. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$

9. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$, $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$

11. See the *Study Guide*.

13. $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$, $[-1 + 2t]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

15. **a.** $\mathcal{B}$ is a basis for V.

b. The coordinate mapping is a linear transformation.

c. The product of a matrix and a vector

d. The coordinate vector of $\mathbf{v}$ relative to $\mathcal{B}$

17. **a.** [**M**]

$$P^{-1} = \frac{1}{32}\begin{bmatrix} 32 & 0 & 16 & 0 & 12 & 0 & 10 \\ & 32 & 0 & 24 & 0 & 20 & 0 \\ & & 16 & 0 & 16 & 0 & 15 \\ & & & 8 & 0 & 10 & 0 \\ & & & & 4 & 0 & 6 \\ & & & & & 2 & 0 \\ & & & & & & 1 \end{bmatrix}$$