Adequacy of the Sheffer Stroke

When dealing with propositional logic we tend to use the logical connectives: $\vee$, $\wedge$, $\to$, $\leftrightarrow$, and $\neg$. Of course these connectives are not all independent from each other. In fact, in "proof system L" we only use $\to$ and $\neg$. All the other connectives are treated as second class citizens — they are abbreviations for statments involving $\to$ and $\neg$. Another way to state this is: "The logical connectives $\to$ and $\neg$ are adequate." Meaning, everything that can be said, could be stated using just $\to$ and $\neg$.

It turns out that we can do even "better". Consider the **Sheffer stroke** denoted by $(\alpha \,|\, \beta)$ whose truth table is:

α	β	$(\alpha \,\vert\, \beta)$
T	T	F
T	F	T
F	T	T
F	F	T

The Sheffer stroke $(P \,|\, Q)$ can be thought of as "not both P and Q."

Assuming that our usual logical connectives, $\{\vee, \wedge, \to, \leftrightarrow, \neg\}$, are adequate, we can reduce this set by noticing that $\alpha \leftrightarrow \beta$ is logically equivalent to $(\alpha \to \beta) \wedge (\beta \to \alpha)$, so $\leftrightarrow$ is redundant. Also, $\alpha \vee \beta$ is logically equivalent to $\neg\alpha \to \beta$ and $\alpha \wedge \beta$ is logically equivalent to $\neg(\alpha \to \neg\beta)$, so we can get along without them as well. Thus $\{\to, \neg\}$ is adequate.

Notice that $\alpha \to \beta$ is equivalent to $\neg\alpha \vee \beta$, so $\{\vee, \neg\}$ is adequate. Or notice that $\alpha \to \beta$ is equivalent to $\neg(\alpha \wedge \neg\beta)$, so $\{\wedge, \neg\}$ is adequate.

On the other hand, $\{\vee, \wedge\}$ is not adequate. Notice that $\alpha \vee \alpha$, $\alpha \wedge \alpha$, and α are all equivalent. There's no hope of building a negation from $\vee$ and $\wedge$ alone!

Consider the following truth tables...

α	$(\alpha \,\vert\, \alpha)$
T	F
F	T

α	$(\neg\alpha)$
T	F
F	T

α	β	$(\alpha \,\vert\, \beta)$	$((\alpha \,\vert\, \beta) \,\vert\, (\alpha \,\vert\, \beta))$
T	T	F	T
T	F	T	F
F	T	T	F
F	F	T	F

α	β	$(\alpha \wedge \beta)$
T	T	T
T	F	F
F	T	F
F	F	F

Thus $(\alpha \,|\, \alpha)$ is equivalent to $(\neg\alpha)$, and $((\alpha \,|\, \beta) \,|\, (\alpha \,|\, \beta))$ is equivalent to $(\alpha \wedge \beta)$. Therefore, since $\{\neg, \wedge\}$ is adequate, any proposition α can be translated to another (logically equivalent) proposition β involving only these two connectives — any occurance of $\neg\sigma$ can be replaced with with $(\sigma \,|\, \sigma)$ and any occurance of $(\sigma \wedge \tau)$ can be replaced with $((\sigma \,|\, \tau) \,|\, (\sigma \,|\, \tau))$. Therefore, $\{\,(\,|\,)\,\}$ is adequate.

Inadequacy of Conjunction and Disjunction Alone

Not that I'm trying to make $\vee$ and $\wedge$ feel bad, but they are inadequate by themselves. O.K. Maybe I do have it out for them. Let's rub it in by proving they are inadequate in a more formal way.

Let β be a proposition logically equivalent to $(\neg\alpha)$ such that β is the shortest proposition built up using only α and the connectives $\wedge$ and $\vee$. This implies that we have the following truth table:

α	$(\neg\alpha)$	β
T	F	F
F	T	T

Since β is built up from α using only the connectives $\wedge$ and $\vee$ (and $\beta = \alpha$ does not yield the correct truth table) we must have that β is either of the form $(\sigma \wedge \tau)$ or $(\sigma \vee \tau)$ where σ and τ are propositions. Clearly σ and τ must be built up from α using only $\wedge$ and $\vee$.

Consider, $(\sigma \wedge \tau)$. If α is F then $\beta = (\sigma \wedge \tau)$ is T thus both σ and τ are T. If α is T then $\beta = (\sigma \wedge \tau)$ is F thus either σ or τ must be F. Thus we have (at least) one of the following:

α	σ
T	F
F	T

OR

α	τ
T	F
F	T

Thus either σ or τ is a proposition logically equivalent to β. But σ and τ are shorter than β and they are built up from α using only $\wedge$ and $\vee$. Thus β is not the shortest such proposition. (contradiction)

Thus we must have that $\beta = (\sigma \vee \tau)$. If α is T then $\beta = (\sigma \vee \tau)$ is F thus both σ and τ are F. If α is F then $\beta = (\sigma \vee \tau)$ is T thus either σ or τ must be T. Thus we have (at least) one of the following:

α	σ
T	F
F	T

OR

α	τ
T	F
F	T

Thus either σ or τ is a proposition logically equivalent to β. But σ and τ are shorter than β and they are built up from α using only $\wedge$ and $\vee$. Again, β is not the shortest such proposition. (contradiction)

Therefore, no such β can exist. Thus $(\neg\alpha)$ cannot be built up from α using only $\wedge$ and $\vee$. Hence, the set $\{\wedge, \vee\}$ is not adequate.