The Sequences $\langle(-1)^n\rangle$ and $\langle\sin(n)\rangle$ Diverge

Theorem $\langle(-1)^n\rangle$ diverges

Proof:

Suppose $(-1)^n \to L$. Let $\epsilon = 1/2 > 0$. There exists some $N \geq 0$ such that

$$|(-1)^n - L| < 1/2 \qquad \text{for all} \qquad n \geq N.$$

In particular, $n = 2N$ and $n = 2N+1$ are both bigger than N so

$$|(-1)^{2N} - L| = |1 - L| < 1/2 \qquad \text{and} \qquad |(-1)^{2N+1} - L| = |-1 - L| = |1 + L| < 1/2.$$

Suppose that $L \geq 0$. Then $|L+1| = 1 + L \geq 1$, but $|1+L| < 1/2$. Therefore, it cannot be the case that $L \geq 0$. Thus we must have that $L < 0$. This means that $-L > 0$ and so $|1-L| = 1 - L > 1$, but $|1-L| < 1/2$. Therefore, $L < 0$ is impossible as well. Thus L must not exist. In other words, the sequence diverges.

This argument can be modified to show that $\langle\sin(n)\rangle$ diverges. The idea is that we if we go down the sequence far enough, we can hit values above $1/2$ and below $-1/2$. So the same argument (with a little finesse) will work.

Theorem $\langle\sin(n)\rangle$ diverges

Proof:

Suppose that $\sin(n) \to L$. Let $\epsilon = 1/4 > 0$. There exists some $N \geq 0$ such that

$$|\sin(n) - L| < 1/4 \qquad \text{for all} \qquad n \geq N$$

. Now let's pick out values for n such that $n \geq N$ and n is as close to $\pi/2 + 2\pi k$ and $3\pi/2 + 2\pi k$ as possible (this is where sin takes on values 1 and -1 respectively). Consider

$$N_1 = \lceil \pi/2 + 2\pi N \rceil \geq \pi/2 + 2\pi N > N \qquad \text{and} \qquad N_2 = \lceil 3\pi/2 + 2\pi N \rceil \geq 3\pi/2 + 2\pi N > N$$

where $\lceil x \rceil$ is the closest integer k such that $k \geq x$ (the "ceiling" function). Notice that $\lceil x \rceil = x + \ell$ for some $0 \leq \ell < 1$. In particular, let $N_1 = \pi/2 + 2\pi N + \ell_1$ and $N_2 = 3\pi/2 + 2\pi N + \ell_2$ where $0 \leq \ell_1, \ell_2 < 1$.

Now $\sin(n)$ decreases on the interval $[\pi/2 + 2\pi N, 3\pi/2 + 2\pi N]$ and increases on the interval $[3\pi/2 + 2\pi N, 5\pi/2 + 2\pi N]$. Thus

$$\sin(N_1) = \sin(\pi/2 + 2\pi N + \ell_1) > \sin(\pi/2 + 2\pi N + 1) = \sin(\pi/2 + 1) \approx 0.54 > 0.5$$

and

$$\sin(N_2) = \sin(3\pi/2 + 2\pi N + \ell_2) < \sin(3\pi/2 + 2\pi N + 1) = \sin(3\pi/2 + 1) \approx -0.54 < -0.5$$

Finally, recalling $N_1, N_2 \geq N$ and that $|\sin(n) - L| < 1/4$ for all $n \geq N$, we have that

$$|\sin(N_1) - L| < 1/4 \qquad \text{and} \qquad |\sin(N_2) - L| < 1/4$$

Suppose that $L \geq 0$. This means that $|\sin(N_2) - L| = -(\sin(N_2) - L) = L - \sin(N_2) > L + 0.5 \geq 0.5$ since $\sin(N_2) < -0.5$. But this is impossible since $|\sin(N_2) - L| < 0.25$. Therefore, it cannot be the case that $L \geq 0$. Thus we must have that $L < 0$. This means that $-L > 0$ and so $|\sin(N_1) - L| = \sin(N_1) - L > 0.5 - L > 0.5$ since $\sin(N_1) > 0.5$. But this cannot be since $|\sin(N_1) - L| < 0.25$. Therefore, $L < 0$ is impossible as well. Thus L must not exist. In other words, the sequence diverges.