Resolution

Proof system L and truth tables aren't the only shows in town. There are many other ways of establishing that propositions are tautologies. Let's look at two different methods below. First, we'll look at "resolution" and then "tableau". Let α be the proposition $\neg(p \vee q) \to (\neg p \vee \neg q)$.

In order to give a resolution proof of α, we must first put $\neg\alpha$ in clausal form (i.e. put $\neg(\neg(p \vee q) \to (\neg p \vee \neg q))$ into CNF (conjuntive normal form) then change notation).

First, we must eliminate the implication "$\to$". We change $\tau_1 \to \tau_2$ into $\neg\tau_1 \vee \tau_2$. Thus $\neg\alpha$ becomes: $\neg(\neg\neg(p \vee q) \vee (\neg p \vee \neg q))$.

Next we eliminate all double negations. $\neg\neg\tau$ becomes τ. Thus we get $\neg((p \vee q) \vee (\neg p \vee \neg q))$. Then we push through negations by replacing $\neg(\tau_1 \vee \tau_2)$ with $\neg\tau_1 \wedge \neg\tau_2$. This gives us: $\neg(p \vee q) \wedge \neg(\neg p \vee \neg q)$. Pushing the negations through again, we get: $(\neg p \wedge \neg q) \wedge (\neg\neg p \wedge \neg\neg q)$. Finally we eliminate double negations one last time and get: $(\neg p \wedge \neg q) \wedge (p \wedge q)$. Ignoring the extra parentheses we have the following CNF for $\neg\alpha$:

$$\neg p \wedge \neg q \wedge p \wedge q$$

So we have four separate clauses (a *clause* is a string of literals and negations of literals connected by disjuctions). Each clause has a single literal. In clausal form $\neg\alpha$ looks like the following:

$$\{\{\neg p\}, \{\neg q\}, \{p\}, \{q\}\}$$

Now we can resolve the two clauses $\{\neg p\}$ and $\{p\}$ and get $\{\} = \Box$. Thus our proof goes as follows: $\{\neg p\}, \{p\}, \Box$. Hence $\neg\alpha$ is unsatisfiable. Thus we have proved α (since $\neg\alpha$ is always false, α must always be true).

Here's another example. Consider the proposition $\beta = (p \to (q \to r)) \to ((p \to q) \to (p \to r))$ which you might recognize as Axiom 3. We need to put $\neg\beta$ into CNF. The first step is to eliminate all of the "$\to$". This leaves us with $\neg\beta = \neg(\neg(\neg p \vee (\neg q \vee r)) \vee (\neg(\neg p \vee q) \vee (\neg p \vee r)))$. Next, all of the negations need to be pushed through and double negations should be eliminated. Here are a few steps of that process: $\neg\neg(\neg p \vee (\neg q \vee r)) \wedge \neg(\neg(\neg p \vee q) \vee (\neg p \vee r))$ then $(\neg p \vee \neg q \vee r) \wedge \neg\neg(\neg p \vee q) \wedge \neg(\neg p \vee r))$ then $(\neg p \vee \neg q \vee r) \wedge (\neg p \vee q) \wedge p \wedge \neg r$. Next, we translate to clausal form:

$$\{\{\neg p, \neg q, r\}, \{\neg p, q\}, \{p\}, \{\neg r\}\}$$

Finally we can write our resolution proof: $\{\neg p, \neg q, r\}$ and $\{\neg r\}$ resolve to $\{\neg p, \neg q\}$. This with $\{p\}$ resolves to $\{\neg q\}$. Next, $\{\neg p, q\}$ with $\{p\}$ resolves to $\{q\}$. And finally, $\{\neg q\}$ and $\{q\}$ resolve to $\{\} = \Box$ (contradiction). Therefore, $\neg\beta$ is unsatisfiable and so β is a tautology.

Proof by Tableau

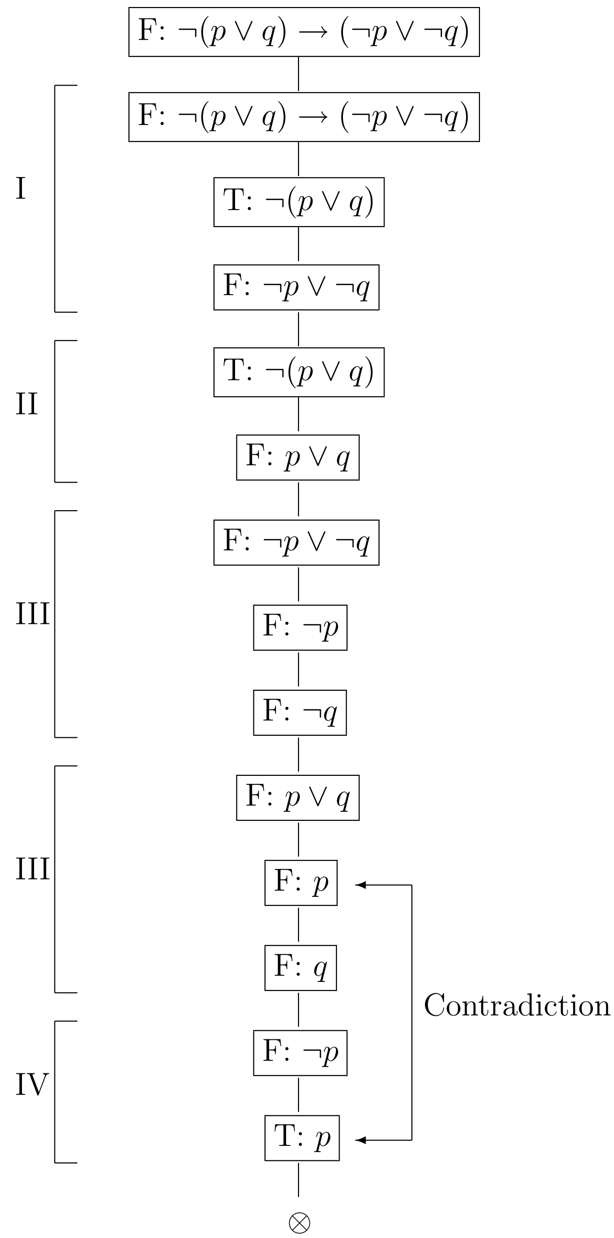

Figure: A tableau proof of $\alpha = \neg(p \lor q) \rightarrow (\neg p \lor \neg q)$.

Explaining the Tableau:

I. For an implication to be false, the hypothesis must be true and the conclusion false.

II. Negation changes true to false.

III. For a disjunction $p \lor q$ to be false, both p and q must be false.

IV. Negation changes false to true.

Our tableau show that assuming the proposition is false leads to contradiction. Therefore, it must be true.

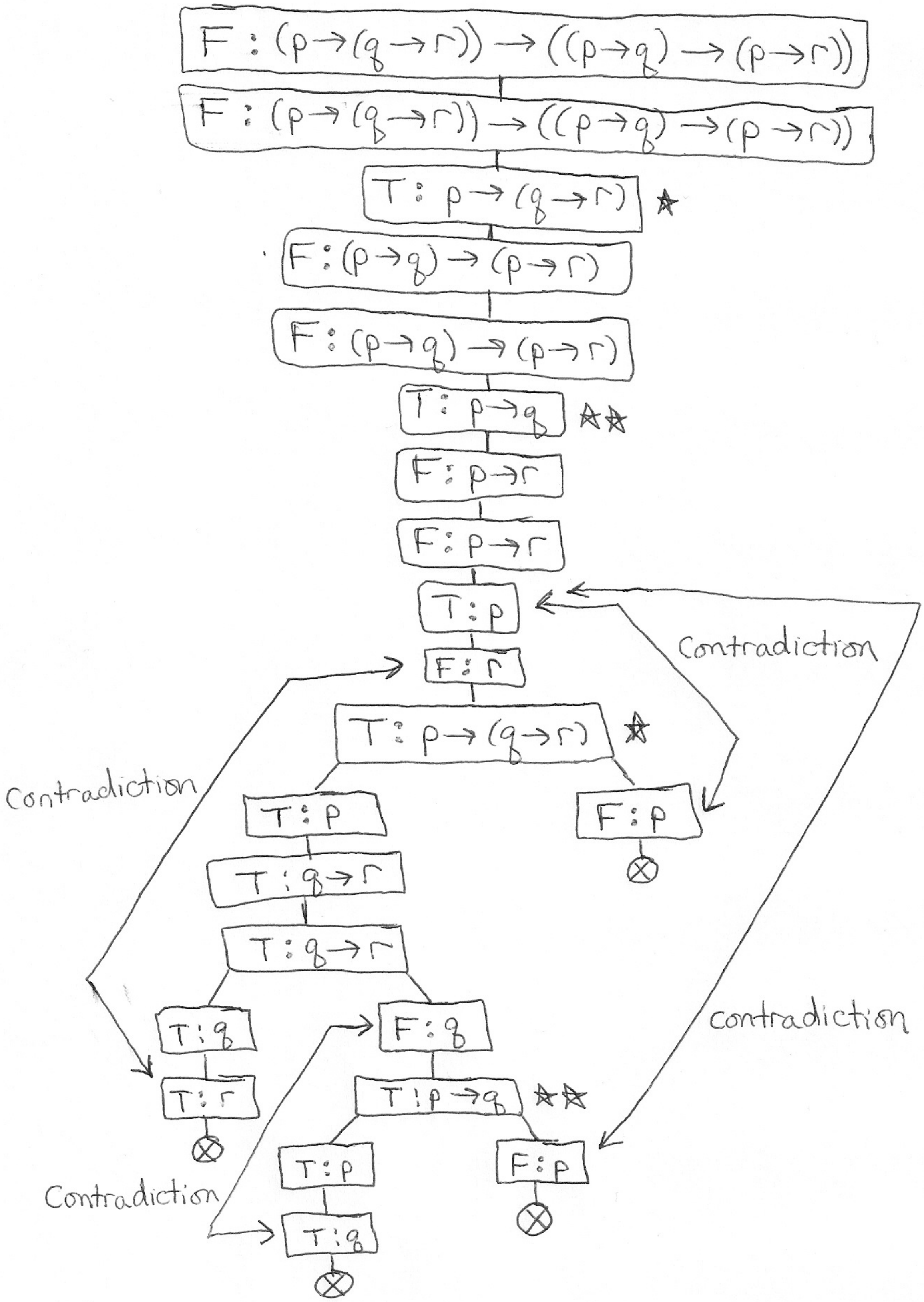

Figure: A tableau proof of $\alpha = (p \to (q \to r)) \to ((p \to q) \to (p \to r))$.