The Cantor-Schröder-Bernstein Theorem

Definition: Let A and B be sets. We say that A and B have the same **cardnality** (denoted $|A| = |B|$ or $\text{card}(A) = \text{card}(B)$) if there exists a bijection (i.e., a one-to-one and onto = invertible function) from A to B.

Notice that...

Reflexive: $|A| = |A|$ (using the identity function),

Symmetric: $|A| = |B|$ implies $|B| = |A|$ (the inverse of a bijective function exists and is bijective itself), and

Transitive: if both $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$ (the composition of two bijective functions is bijective).

These facts tell us that the relation $|A| = |B|$ is an **equivalence relation**.

If cardinality is to measure *size*, we should be able to compare different sizes with some kind of $\leq$ relation.

Definition: We say $|A| \leq |B|$ if there exists an injection (a one-to-one function) from A to B.

One can show that there is an injection from A to B if and only if there is a surjection from B to A. We will skip the proof. However, keeping this fact in mind, we could replace the following discussions built on one-to-one functions with discussions built on onto functions.

Notice that...

Reflexive: $|A| \leq |A|$ (using the identity function),

Anti-Symmetric: if both $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$ (this requires proof), and

Transitive: $|A| \leq |B|$ and $|B| \leq |C|$ implies that $|A| \leq |C|$ (the composition of two injective functions is injective).

These facts tell us that the relation $|A| \leq |B|$ is a **partial order**.

Notice that we didn't supply a quick justification of anti-symmetry. It turns out that this is a little tricky to establish (Cantor didn't immediately see how to prove this). In fact, this result is called the Cantor-Schröder-Bernstein theorem.

Theorem: Let A and B be sets such that $|A| \leq |B|$ and $|B| \leq |A|$. Then $|A| = |B|$.

Proof: Since $|A| \leq |B|$ there exists an injection $f : A \to B$ and since $|B| \leq |A|$ there exists an injection $g : B \to A$. Let's define the follow sets inductively. First, let $S_1 = A - g(B)$. Then for each $n \geq 1$, let $S_{n+1} = g(f(S_n))$. Finally let $S = \displaystyle\bigcup_{n=1}^{\infty} S_n$. We can now define our candidate bijection. Let $h : A \to B$ be defined by

$$h(x) = \begin{cases} f(x) & x \in S \\ g^{-1}(x) & x \in A - S \end{cases}$$

First, notice that if $x \in A - S$, then $x \notin S$. So $x \notin S_n$ for all $n \geq 1$. In particular, $x \notin S_1 = A - g(B)$. Therefore, $x \in g(B)$, so x is the image of something in B. But g is injective, so x is the image of exactly one element of B. We call this (unique) element "$g^{-1}(x)$". Therefore, h is a well-defined function.

Let's show h is injective. Suppose that $h(x) = h(y)$. There are four possibilities: (1) $x, y \in S$, (2) $x, y \in A - S$, (3) $x \in S$ and $y \in A - S$, and (4) $x \in A - S$ and $y \in S$. In case (1): $f(x) = h(x) = h(y) = f(y)$ thus $x = y$ because f is injective. Case (2): $g^{-1}(x) = h(x) = h(y) = g^{-1}(y)$. Note that $z = g^{-1}(x) = g^{-1}(y)$ is the element which maps to x and maps to y. Now g is well-defined, so these better be the same. Thus $x = y$. Finally, cases (3) and (4) are symmetric, so without loss of generality, let's assume that $x \in S$ and $y \in A - S$. Thus $f(x) = h(x) = h(y) = g^{-1}(y)$ so that $g(f(x)) = y$. Now $x \in S$ so $x \in S_m$ for some $m \geq 1$. Then $y = g(f(x)) \in g(f(S_m)) = S_{m+1}$. Therefore, $y \in S$ which is a contradiction. Therefore, cases (3) and (4) cannot actually occur. Thus in all cases we have $x = y$. Therefore, h is one-to-one.

We also need to show that h is surjective (i.e., onto). Suppose that $y \in B$. Either $y \in f(S)$ or $y \notin f(S)$. If $y \in f(S)$, then $y = f(x)$ for some $x \in S$. Now $x \in S$, so $h(x) = f(x) = y$. Therefore, y is in the range of h.

Next, consider $y \notin f(S)$. Then $g(y) \in g(B)$ so $g(y) \notin A - g(B) = S_1$. Could $g(y)$ be in $S_{n+1} = g(f(S_n))$ for $n > 1$? If so, $g(y) = g(z)$ for some $z \in f(S_n)$. However, g is one-to-one, so $g(y) = g(z)$ implies $y = z \in f(S_n)$. But since S_n is contained in S, this means $y \in f(S)$ (contradicting our assumption). So $g(y)$ cannot belong to any S_n and thus $g(y) \notin S$. We now have $g(y) \in A - S$ and so $h(g(y)) = g^{-1}(g(y)) = y$. Once again, this means y is in the range of h. Therefore, in both cases (i.e., $y \in f(S)$ and $y \notin f(S)$), we have y is in the range of h. Thus h is surjective.

We have shown that h is injective and surjective. Thus h is bijective, so there exists a bijection from A to B. Therefore, $|A| = |B|$. ■