Appendix #22 Negate the following: "For any real number c, $x < y \Rightarrow cx < cy$."

Answer: "There exists a real number c such that $x < y$ does not imply $cx < cy$." or "There exists real numbers c, x, and y where $x < y$ and $cx \geq cy$."

[Notice that the original statement is false. Whereas our negated statement is true. Simply let $c = -1$, $x = 1$, and $y = 2$. Then $x < y$ but $cx > cy$.]

1.1 #18 Show that $A \subseteq B$ if and only if $B' \subseteq A'$.

Here are two different approaches. First, the "standard" way.

Proof #1 Assume $A \subseteq B$. Suppose $x \in B'$ (we need to show that $x \in A'$). Since $x \in B'$, $x \notin B$. Suppose $x \in A$, then $x \in B$ since $A \subseteq B$. But this is impossible since $x \notin B$. Therefore, x cannot be a member of A. Thus $x \notin A$ so that $x \in A'$. Thus we have shown that $B' \subseteq A'$.

Now assume $B' \subseteq A'$. [Essentially repeat the same proof, but change A into B' and so forth.] Suppose $x \in A$. Now if $x \notin B$, then $x \in B'$ and hence $x \in A'$ (since $B' \subseteq A'$). Therefore, $x \notin A$. But this is impossible since $x \in A$. Thus x must be a member of B (it cannot not be a member since this leads to a contradiction). So we have shown $A \subseteq B$.

Proof #2 If you notice this statement is a "set theoretic" version of the logical equivalence of an implication and its contrapositive, you can get the following quick-slick proof:
$A \subseteq B \Leftrightarrow \forall x, (x \in A \to x \in B) \Leftrightarrow \forall x, (x \notin B \to x \notin A) \Leftrightarrow \forall x, (x \in B' \to x \in A') \Leftrightarrow B' \subseteq A'$

1.2 #9(b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = \begin{cases} \frac{x}{2} & x \text{ is even} \\ \frac{x-3}{2} & x \text{ is odd} \end{cases}$ Determine if f is one-to-one, onto, both, or neither. Prove your answer.

To determine whether f is one-to-one or onto, let's start with some numerical experiments.

$x =$	$f(x) =$
-2	-1
-1	-2
0	0
1	-1
2	1
3	0

From the table is *seems* that f is onto, and we already have a counterexample to show that f is not one-to-one.

f is not one-to-one.
proof: $f(-2) = -1 = f(1)$.

To show that f is onto, notice that both "pieces" of f are onto, so we can use either part. Let's use the "even" part of the function. We want to find x so that $f(x) = y$. Also, we want to use the even part of the function so we need x to be an even integer. Solving $f(x) = x/2 = y$ for x, we get $x = 2y$ (which is automatically even!)

f is onto.
proof: Suppose $y \in \mathbb{Z}$. Then $2y$ is an even integer so that $f(2y) = (2y)/2 = y$. Thus y is in the range of f. Therefore, f is onto.

1.2 #9(d) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = \begin{cases} 2x - 1 & x \text{ is even} \\ 2x & x \text{ is odd} \end{cases}$ Determine if f is one-to-one, onto, both, or neither. Prove your answer.

To determine whether f is one-to-one or onto, let's start with some numerical experiments.

$x =$	$f(x) =$
-2	-5
-1	-2
0	-1
1	2
2	3
3	6

From the table is *seems* that f is one-to-one, but not onto. In particular, it looks like 0 is not in the range of f.

To show that f is one-to-one we will have to consider several different cases (since our function is piecewise defined).

f is one-to-one.
proof: Suppose that $f(x) = f(y)$. Consider the following cases:

Both x and y are even In this case, $2x - 1 = f(x) = f(y) = 2y - 1$ so $2x = 2y$ and thus $x = y$.

Both x and y are odd In this case, $2x = f(x) = f(y) = 2y$ so $x = y$.

One even, one odd Without loss of generality (WLOG) assume x is even and y is odd. Then $2x - 1 = f(x) = f(y) = 2y$. But this is impossible since $2x - 1$ must be odd and $2y$ must be even (we cannot have "EVEN=ODD"). Since we have reached a contradiction, this case does not occur.

In all (non-vacuous) cases $x = y$. Therefore, f is one-to-one.

f is not onto.
proof: We will show 0 is not in the range of f. If it were, we would have $f(x) = 0$ for some x. If x were even, then $2x - 1 = f(x) = 0$. So that $2x = 1$ and thus $x = 1/2$ (not an integer). So x cannot be even — it must be odd. Thus $2x = f(x) = 0$ and so $x = 0$, but 0 is not odd. Therefore, x is neither even nor odd — it does not exist! Therefore, nothing maps to 0 and so f is not onto.