Math 3110-101 **Homework #3** **Due: Mon., Sept. 14^{th}, 2020**

1. A function problem

   (a) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x^2 - 3$.

      i. Show f is not 1-1.

      ii. Show f is not onto.

      iii. Let $A = \{-1, 0, 1, 2, 3\}$. Find $f(A) = \{f(x) \mid x \in A\}$ (the image of the set A under the map f).

      iv. Let $A = \{-1, 0, 1, 2, 3\}$. Find $f^{-1}(A) = \{x \in \mathbb{Z} \mid f(x) \in A\}$ (the inverse image of A).

   (b) Let $g : X \to Y$. Prove that g is onto if and only if $g^{-1}(B) \neq \phi$ (the inverse image of B is non-empty) for all non-empty subsets of Y: $\phi \neq B \subset Y$.

   Recall that for $A \subseteq X$ and $B \subseteq Y$ ...

$$f(A) = \{f(x) \mid x \in A\} \subseteq Y \qquad \text{and} \qquad f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X$$

2. Dihedral groups: generators and relations style. Recall that ...

$$D_4 = \langle x, y \mid x^4 = 1,\ y^2 = 1,\ \text{and } (xy)^2 = 1 \rangle = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$$

   (a) Write down the Cayley table for D_4.

   (b) Find the inverse of each element (i.e. $1^{-1} =???$, $x^{-1} =???$, etc.).

   (c) Find the order of each element.

   (d) Find all of the **distinct** cyclic subgroups of D_4.

   (e) What is in $Z(D_4)$ (recall that $Z(G)$ is the *center* of G)?

   (f) Simplify $x^6y^{-3}x^3y^8x^{-5}yxy$.

3. The Matrix problem

   (a) Compute $A^{-1}B^2$ where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}_9)$

   (b) Find the cyclic subgroup generated by A. What is the order of A?

4. Let H and K be subgroups of a group G. Show that $H \cap K$ (the intersection of H and K) is a subgroup of G.