**Math 3110-101** # Homework #6 **Due: Wed., Oct. $21^{\text{st}}$, 2020**

1. Let $H$ and $K$ be subgroups of $G$.
   (a) Suppose that $H$ and $K$ are normal subgroups of $G$. Show that $H \cap K$ is a normal subgroup of $G$ as well.
   *Note:* Please include a careful proof that $H \cap K$ is a subgroup – even though we've shown this before.
   (b) Let $|G| = 36$, $|H| = 12$, and $|K| = 18$. Using Lagrange's Theorem, what are the possible orders of $H \cap K$?

2. Let $H = \{1, x^3, x^6\} \subseteq D_9 = \{1, x, \dots, x^8, y, xy, \dots, x^8y\} = \langle x, y \mid x^9 = 1, y^2 = 1, xyxy = 1 \rangle$. Notice $H = \langle x^3 \rangle$, so $H$ is a subgroup of $D_9$. Quickly compute $[D_9 : H]$ (i.e. the index of $H$ in $D_9$). Then find all of the left and right cosets of $H$ in $D_9$. Is $H$ a normal subgroup of $D_9$?

3. Direct products of cyclic groups.[1]
   (a) Find the order of $(5, 2, 6)$ in $\mathbb{Z}_6 \times \mathbb{Z}_8 \times \mathbb{Z}_{99}$.
   (b) Explain why $\mathbb{Z}_{20} \cong \mathbb{Z}_4 \times \mathbb{Z}_5$ but $\mathbb{Z}_{20} \not\cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$.
   In addition, list the distinct orders of elements in both $\mathbb{Z}_4 \times \mathbb{Z}_5$ and $\mathbb{Z}_2 \times \mathbb{Z}_{10}$ and give an example of an element of each such order.

4. Let $G$ and $H$ be groups.
   (a) Show $\{e\} \times H = \{(e, h) \mid h \in H\}$ is a normal subgroup of $G \times H$ (where $e$ is the identity of $G$).
   *Note:* You need to show that $\{e\} \times H$ is a subgroup AND that it's normal.
   (b) Show $G \times H \cong H \times G$.

[1] Gallian uses $G \oplus H$ for direct products. I will use the more standard $G \times H$ notation.