Math 3110-101

Homework #7

Due: Wed., Oct. 28^{th}, 2020

1. Recall that the center of a group is a normal subgroup. $Z = Z(D_6) = \{1, x^3\}$ where $D_6 = \{1, x, \ldots, x^5, y, xy, \ldots, x^5y\}$. Find the distinct cosets of Z in D_6 then write down a Cayley table for D_6/Z. Is D_6/Z abelian? Is it cyclic? Explain your answer.

2. Let $\varphi : D_n \to \{\pm 1\}$ be defined by $\varphi(x) = \begin{cases} +1 & x \text{ is a rotation} \\ -1 & x \text{ is a reflection} \end{cases}$

   Show that φ is a homomorphism. What is the kernel of φ? What does the first isomorphism theorem tell us here?

3. Quotients in $\mathbb{Z}_n$.

   (a) Let $H = \langle 5 \rangle \subseteq \mathbb{Z}_{100}$. First $H = \{???\}$. Then compute the cosets of H in $\mathbb{Z}_{100}$. Write down a Cayley table for $\mathbb{Z}_{100}/H$. What familiar group is this isomorphic to?

   (b) Let k, ℓ, n be positive integers such that $n = k\ell$ (i.e. k divides n). Make a conjecture about what the quotient $\mathbb{Z}_n/\langle k \rangle$ is isomorphic to. Then prove your conjecture.

   Hint: Define the map $\varphi(x) = x$ from $\mathbb{Z}_n$ to your target group and then use the first isomorphism theorem. Don't forget to show that φ is a well-defined homomorphism.

4. Let G be a finite group, H a normal subgroup of G, and $g \in G$. Show that $|gH|$ divides $|g|$ (in G).

   Note: In this problem, $|gH|$ means the order of the element gH in the quotient group G/H (as opposed to the cardinality of gH as a set).

   Unnecessary Note: The assumption that G is finite is totally unnecessary if one uses the standard convention that all positive integers divide infinity.

5. Let G and H be finite groups and let $\varphi : G \to H$ be an epimorphism (this is a homomorphism which is onto). Suppose that there is some $x \in H$ such that $|x| = 8$. Prove that G has an element of order 8 as well.