Math 3110-101 # Homework #8 **Due: Fri., Nov. 13^{th}, 2020**

1. In each of the following rings, R, state the characteristic of the ring. If R has unity, give an example of a unit and its inverse (other than 1 itself). If no unit exists, explain why not. If R has zero divisors, give an example of a zero divisor. If no zero divisors exist, explain why not.
   (a) $R = \mathbb{Z}_9 \times \mathbb{Z}_{12}$
   (b) $R = 4\mathbb{Z} = \{4k \mid k \in \mathbb{Z}\}$ (multiples of 4)
   (c) $R = (\mathbb{Z}_5)^{2\times 2}$ (2×2 matrices with entries in $\mathbb{Z}_5$)
2. Recall $R \times S$ is the direct product of the rings R and S.
   (a) Suppose R and S have 1's. Then show $R \times S$ is also a ring with 1.
   (b) Let R and S be non-trivial rings (i.e. not the zero ring). Show that $R \times S$ is never an integral domain.
3. Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}$.
   We define the *norm* of $z = a+b\sqrt{-3} \in R$ by $N(z) = z\bar{z} = (a+b\sqrt{-3})(a-b\sqrt{-3}) = a^2+3b^2$. Note: $N(z) \in \mathbb{Z}_{\geq 0}$.
   (a) For all $z, w \in R$, prove that $N(zw) = N(z)N(w)$.
   (b) When is $N(z) = 0$? When is $N(z) = 1$?
   (c) If $u \in U(R)$, then what can be said about $N(u)$? [*Hint:* Consider $N(uu^{-1})$.] Determine $U(R)$.
   (d) Prove that R is an integral domain. [*Hint:* Use the subring test to show R is a subring of $\mathbb{C}$. Also, although there are other ways to prove this, please use the norm to show R has no zero divisors.]
4. Prove that $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field. [Use a subring test to show it is a ring. Use a "conjugate trick" to compute the inverse of a non-zero element.]