Homework #9

Due: Fri., Nov. 20^{th}, 2020

#1 Units and Zero Divisors Let R be a ring with 1. Recall that $R^\times = \{u \in R \mid u^{-1} \text{ exists}\}$ is the group of units. Also, if $a, b \in R$ such that $a \neq 0$, $b \neq 0$ but $ab = 0$, then we call a a *left zero divisor* and b a *right zero divisor*. A *zero divisor* is either a left or right zero divisor (or both).

(a) Let $u \in R^\times$. Show that $-u \in R^\times$.

(b) Suppose that $r \in R$ is a zero divisor. Show that r is not a unit.

Note: You need to consider two cases. One case where r is a left zero divisor and another where r is a right zero divisor.

#2 Sticking to my Principals Let R be a commutative ring with 1 and let $a \in R$. Recall that $(a) = \{ra \mid r \in R\}$ is the principal ideal generated by a.

(a) List all of the (distinct) principal ideals in $\mathbb{Z}_{12}$. Show their contents. Example: $(0) = \{0\}$.

(b) Let R be an integral domain. Show that $(a) = (b)$ if and only if a and b are associates (i.e., $a = ub$ for some unit $u \in R^\times$).

#3 Ideal Arithmetic Let R be ring and $I, J \lhd R$. One defines $I + J = \{x + y \mid x \in I \text{ and } y \in J\}$ to be the *sum* of I and J. Similarly, one defines $IJ = \{x_1y_1 + \cdots + x_\ell y_\ell \mid \text{for some } \ell \geq 0 \text{ and } x_1, \ldots, x_\ell \in I \text{ and } y_1, \ldots, y_\ell \in J\}$ to be the *product* of I and J. It can be shown that $I + J$, IJ, and $I \cap J$ are ideals of R.

(a) Prove that $I + J \lhd R$.

(b) Consider $I = (12)$ and $J = (15)$ in $\mathbb{Z}$. Calculate $I + J$, IJ, and $I \cap J$.

Since $\mathbb{Z}$ is a principal ideal domain, $(a)(b) = (c)$, $(a) + (b) = (d)$, and $(a) \cap (b) = (\ell)$ for some $c, d, \ell \in \mathbb{Z}$. Make a conjecture about how a and b are related to c, d, and ℓ.

Note: You don't have to prove your conjectures. Just tell me how you think this "ideal arithmetic" works.

#4 Prime and Maximal As a quick reminder, in $\mathbb{Z}$ and in $\mathbb{Z}_n$, we know that subgroup = normal subgroup = cyclic subgroup = subring = ideal = principal ideal.

(a) Find all the ideals of $\mathbb{Z}_{24}$ and draw the corresponding lattice. Which ideals are prime? Which are maximal?

(b) Determine which ideals in $\mathbb{Z}$ are prime and which are maximal. [Prove your assertions.]

Note: Don't forget to consider the trivial ideal: $\{0\}$.