Math 3110-101

Homework #7

Due: Wed., Oct. 25th, 2023

Please remember when submitting any work via email or in person to...

PUT YOUR NAME ON YOUR WORK!

#1 Let H and K be subgroups of G.

(a) Suppose that H and K are normal subgroups of G. Show that $H \cap K$ is a normal subgroup of G as well. *Note:* Please include a careful proof that $H \cap K$ is a subgroup – even though we've shown this before.

(b) Let $|G| = 100$, $|H| = 50$, and $|K| = 20$. Using Lagrange's Theorem, what are the possible orders of $H \cap K$?

#2 Let $H = \{1, x^4, y, x^4y\} \subseteq D_8 = \{1, x, \ldots, x^7, y, xy, \ldots, x^7y\} = \langle x, y \mid x^8 = 1, y^2 = 1, xyxy = 1\rangle$. It isn't hard to show that H is closed under the operation in D_8, thus by the finite subgroup test, H is a subgroup of D_8.

Quickly compute $[D_8 : H]$ (i.e. the index of H in D_8). Then find all of the left and right cosets of H in D_8. Is H a normal subgroup of D_8?

#3 Direct products of cyclic groups.[1]

(a) Find the order of $(5, 6, 44)$ in $\mathbb{Z}_9 \times \mathbb{Z}_8 \times \mathbb{Z}_{66}$.

(b) Explain why $\mathbb{Z}_{50} \cong \mathbb{Z}_2 \times \mathbb{Z}_{25}$ but $\mathbb{Z}_{50} \not\cong \mathbb{Z}_5 \times \mathbb{Z}_{10}$. In addition, list the distinct orders of elements in both $\mathbb{Z}_2 \times \mathbb{Z}_{25}$ and $\mathbb{Z}_5 \times \mathbb{Z}_{10}$ and give an example of an element of each such order.

#4 Let G and H be groups.

(a) Show $\{e\} \times H = \{(e, h) \mid h \in H\}$ is a normal subgroup of $G \times H$ (where e is the identity of G). *Note:* You need to show that $\{e\} \times H$ is a subgroup AND that it's normal.

(b) Show $G \times H \cong H \times G$.

[1] Gallian uses $G \oplus H$ for direct products. I will use the more standard $G \times H$ notation.