Math 3110-101 # Homework #9 **Due: Fri., Nov. 17th, 2023**

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PUT YOUR NAME ON YOUR WORK!

#1 In each of the following rings, R, state the characteristic of the ring. If R has unity, give an example of a unit and its inverse (other than 1 itself). If no unit exists, explain why not. If R has zero divisors, give an example of a zero divisor. If no zero divisors exist, explain why not.

(a) $R = \mathbb{Z}_{10} \times \mathbb{Z}_{15}$

(b) $R = 99\mathbb{Z} = \{99k \,|\, k \in \mathbb{Z}\}$ (multiples of 99)

(c) $R = (\mathbb{Z}_7)^{2\times 2}$ (2×2 matrices with entries in $\mathbb{Z}_7$)

#2 Let R be a ring with 1. Let u be a unit and z be a zero divisor. Prove that both uz and zu are zero divisors.

Note: Argue carefully. You will need to write separate arguments for the cases where z is a left zero divisor and where z is a right zero divisor.

#3 Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ where $i^2 = -1$ (i.e., $i = \sqrt{-1}$).

This is the ring of Gaussian integers. This ring is equipped with an extra tool called an algebraic *norm*.

We define the normal of $z = a + bi$ by $N(z) = z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$. Notice that $N(z) \in \mathbb{Z}_{\geq 0}$.

A very helpful observation is that for any $w, z \in \mathbb{Z}[i]$, we have $N(wz) = N(w)N(z)$.

(a) Prove $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$.

(b) When is $N(z) = 0$? When is $N(z) = 1$?

(c) If $u \in U(\mathbb{Z}[i])$, then what can be said about $N(u)$? [*Hint:* Consider $N(uu^{-1})$.] Determine $U(\mathbb{Z}[i])$.

(d) Prove that $\mathbb{Z}[i]$ has no zero divisors. [*Hint:* Use the norm.]

Note: Since $\mathbb{Z}[i]$ is a commutative ring with $1 \neq 0$ and it has no zero divisors, it is an *integral domain.*

#4 Prove that $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$ is a field.

Note: Use a subring test to show it is a ring (it is a subring of $\mathbb{R}$).

Then use the "conjugate trick" to show it has multiplicative inverses.